

The definition of a vector space $(V, +, \cdot)$

1. For any \vec{u} and \vec{v} in V , $\vec{u} + \vec{v}$ is also in V .
2. For any \vec{u} and \vec{v} in V , $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. For any \vec{u} , \vec{v} , \vec{w} in V , $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
4. There is an element in V called the *zero* or *null* vector, which we denote by $\vec{0}$, such that for all \vec{u} in V we have $\vec{0} + \vec{u} = \vec{u}$.
5. For every \vec{u} in V , there is a vector called the *negative* of \vec{u} and denoted $-\vec{u}$, such that $-\vec{u} + \vec{u} = \vec{0}$.
6. If k is any scalar in \mathbb{R} and \vec{u} is any vector in V , then $k \cdot \vec{u}$ is a vector in V .
7. For any scalar k in \mathbb{R} and any vectors \vec{u} and \vec{v} in V ,
 $k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$.
8. For any scalars k and m in \mathbb{R} and any vector \vec{u} in V ,
 $(k + m) \cdot \vec{u} = k \cdot \vec{u} + m \cdot \vec{u}$.
9. For any scalars k and m in \mathbb{R} and any vector \vec{u} in V ,
 $k \cdot (m \cdot \vec{u}) = (k m) \cdot \vec{u}$.
10. For any vector \vec{u} in V , $1 \cdot \vec{u} = \vec{u}$.

What determines a vector space?

- A nonempty set V whose elements are called *vectors*.
- An operation $+$ called vectors addition, such that

$$\text{vector} + \text{vector} = \text{vector}$$

In other words, we have *closure* under vector addition.

- An operation \cdot called scalar multiplication, such that

$$\text{scalar} \cdot \text{vector} = \text{vector}$$

In other words, we have *closure* under scalar multiplication.

- The remaining 8 axioms are all satisfied.

Some basic identities in a vector space

Theorem: Let V be a vector space. The following statements are always true.

a) $0 \cdot \vec{u} = \vec{0}$

b) $k \cdot \vec{0} = \vec{0}$

c) $(-1) \cdot \vec{u} = -\vec{u}$

d) If $k \cdot \vec{u} = \vec{0}$ then $k = 0$ or $\vec{u} = \vec{0}$.

Vector subspace

Let $(V, +, \cdot)$ be a vector space.

Definition: A subset W of V is called a *subspace* if W is itself a vector space with the operations $+$ and \cdot defined on V .

Theorem: Let W be a nonempty subset of V . Then W is a subspace of V if and only if it is *closed* under addition and scalar multiplication, in other words, if:

- i.) For any \vec{u}, \vec{v} in W , we have $\vec{u} + \vec{v}$ is in W .
- ii.) For any scalar k and any vector \vec{u} in W we have $k \cdot \vec{u}$ is in W .

Linear combinations and the span of a set of vectors

Let $(V, +, \cdot)$ be a vector space.

Definition: Let \vec{v} be a vector in V . We say that \vec{v} is a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$ if there are scalars k_1, k_2, \dots, k_r such that

$$\vec{v} = k_1 \cdot \vec{v}_1 + k_2 \cdot \vec{v}_2 + \dots + k_r \cdot \vec{v}_r$$

The scalars k_1, k_2, \dots, k_r are called the *coefficients* of the linear combination.

Definition: Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ be a set of vectors in V . Let W be the set of **all linear combinations** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$:

$$W = \{k_1 \cdot \vec{v}_1 + k_2 \cdot \vec{v}_2 + \dots + k_r \cdot \vec{v}_r : \text{for all choices of scalars } k_1, k_2, \dots, k_r\}$$

Then W is called the *span* of the set S .

We write:

$$W = \text{span } S \quad \text{or} \quad W = \text{span } \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$$

Linear combinations and the span of a set of vectors

Let $(V, +, \cdot)$ be a vector space and let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ be a set of vectors in V .

$\text{span} S = \{k_1 \cdot \vec{v}_1 + k_2 \cdot \vec{v}_2 + \dots + k_r \cdot \vec{v}_r : \text{for all scalars } k_1, k_2, \dots, k_r\}$

(all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r$).

Theorem: $\text{span} \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a subspace of V .

It is in fact the *smallest* subspace of V that contains all vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$.

Linear independence in a vector space V

Let V be a vector space and let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ be a set of vectors in V .

Definition: The set S is called *linearly independent* if the vector equation

$$(*) \quad c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \dots + c_r \cdot \vec{v}_r = \vec{0}$$

has **only one** solution, the trivial one:

$$c_1 = 0, \quad c_2 = 0, \quad \dots, \quad c_r = 0$$

The set is called linearly dependent otherwise, if equation (*) has other solutions besides the trivial one.

Theorem: The set of vectors S is linearly independent if and only if **no** vector in the set is a linear combination of the other vectors in the set.

The set of vectors S is linearly dependent if and only if one of the vectors in the set is a linear combination of the other vectors in the set.

The solutions to a homogeneous system of equations

$$A \cdot \vec{x} = \vec{0}$$

Consider a *homogeneous* system of m equations with n unknowns.

In other words,

- let A be an $m \times n$ matrix
- let \vec{x} be an $n \times 1$ matrix (or vector) whose entries are the unknowns x_1, x_2, \dots, x_n
- let $\vec{0}$ denote the $n \times 1$ matrix (vector) whose entries are all 0.

The system can then be written as

$$A \cdot \vec{x} = \vec{0}$$

Theorem: The set of solutions to a *homogeneous* system of m equations with n unknowns is a **subspace** of \mathbb{R}^n .

Reminder from MA1201 on systems of equations

The case when $\# \text{ of equations} = \# \text{ of unknowns}$

Theorem: Let A be a **square** matrix in \mathbf{M}_{nn} .

The following statements are equivalent:

1. A is invertible
2. $\det(A) \neq 0$
3. The homogeneous system $A \cdot \vec{x} = \vec{0}$ has only the trivial solution
4. The system of equations $A \cdot \vec{x} = \vec{b}$ has exactly one solution for every vector \vec{b} in \mathbb{R}^n .

The case when $\# \text{ of unknowns} > \# \text{ of equations}$

Theorem: Let A be a matrix in \mathbf{M}_{mn} , where $n > m$.

Then the homogeneous system $A \cdot \vec{x} = \vec{0}$ has infinitely many solutions.

Basis in a vector space V

Definition: A set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ of vectors is called a *basis* for V if

1. S is linearly independent
2. $\text{span}(S) = V$.

In other words, the set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ is a basis for V if

1. The equation $c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \dots + c_r \cdot \vec{v}_r = \vec{0}$ has only the trivial solution.
2. The equation $c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \dots + c_r \cdot \vec{v}_r = \vec{b}$ has a solution for every \vec{b} in V .

Standard bases for the most popular vector spaces

- In \mathbb{R}^2 : $\{\vec{i}, \vec{j}\}$.
- In \mathbb{R}^3 : $\{\vec{i}, \vec{j}, \vec{k}\}$.
- In \mathbb{R}^n : $\{e_1, e_2, \dots, e_n\}$.
- In \mathbf{P}_n : $1, X, X^2, \dots, X^n$.
- In \mathbf{M}_{22} : all matrices with all entries 0 except for one entry, which is 1. There are 4 such matrices.
- In \mathbf{M}_{mn} : all matrices with all entries 0 except for one entry, which is 1. There are $m \cdot n$ such matrices.

Dimension of a vector space

Some vector spaces do not have a *finite* basis.

A vector space has many different bases. However,

Theorem: All bases of a finite dimensional vector space have the **same number** of elements.

Definition: Let V be a finite dimensional vector space. We call *dimension* of V is the number of elements of a basis for V . We use the notation $\dim(V)$ for the dimension of V .

Example: Counting the elements of the standard basis of each of the popular vectors spaces, we have that:

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbf{P}_n) = n + 1$
- $\dim(\mathbf{M}_{m \times n}) = m \cdot n$

Finding bases for the null space, row space and column space of a matrix

Given an $m \times n$ matrix A

1. Reduce the matrix A to the *reduced row echelon form* R .
2. Solve the system $R \cdot \vec{x} = \vec{0}$. Find a basis for the solutions space.

The **same** basis for the solution space of $R \cdot \vec{x} = \vec{0}$ is a basis for the null space of A .

3. Consider the non-zero rows of R . They form a basis for the row space of R .

The **same** basis for the row space of R is a basis for the row space of A .

4. Take the columns of R with leading 1s. They form a basis for the column space of R .

The **corresponding** column vectors in A form a basis for the column space of A .