# The definition of a vector space $(V, +, \cdot)$

1. For any $\vec{u}$ and $\vec{v}$ in $V$, $\vec{u} + \vec{v}$ is also in $V$.
2. For any $\vec{u}$ and $\vec{v}$ in $V$, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. For any $\vec{u}$, $\vec{v}$, $\vec{w}$ in $V$, $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
4. There is an element in $V$ called the *zero* or *null* vector, which we denote by $\vec{0}$, such that for all $\vec{u}$ in $V$ we have $\vec{0} + \vec{u} = \vec{u}$.
5. For every $\vec{u}$ in $V$, there is a vector called the *negative* of $\vec{u}$ and denoted $-\vec{u}$, such that $-\vec{u} + \vec{u} = \vec{0}$.
6. If $k$ is any scalar in $\mathbb{R}$ and $\vec{u}$ is any vector in $V$, then $k \cdot \vec{u}$ is a vector in $V$.
7. For any scalar $k$ in $\mathbb{R}$ and any vectors $\vec{u}$ and $\vec{v}$ in $V$, $k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$.
8. For any scalars $k$ and $m$ in $\mathbb{R}$ and any vector $\vec{u}$ in $V$, $(k + m) \cdot \vec{u} = k \cdot \vec{u} + m \cdot \vec{u}$.
9. For any scalars $k$ and $m$ in $\mathbb{R}$ and any vector $\vec{u}$ in $V$, $k \cdot (m \cdot \vec{u}) = (k\,m) \cdot \vec{u}$.
10. For any vector $\vec{u}$ in $V$, $1 \cdot \vec{u} = \vec{u}$.

# What determines a vector space?

- A nonempty set $V$ whose elements are called *vectors*.
- An operation $+$ called vectors addition, such that

  $$\text{vector} \; + \; \text{vector} \; = \; \text{vector}$$

  In other words, we have *closure* under vector addition.
- An operation $\cdot$ called scalar multiplication, such that

  $$\text{scalar} \; \cdot \; \text{vector} \; = \; \text{vector}$$

  In other words, we have *closure* under scalar multiplication.
- The remaining 8 axioms are all satisfied.

# Some basic identities in a vector space

Theorem: Let $V$ be a vector space. The following statements are always true.

a) $0 \cdot \vec{u} = \vec{0}$
b) $k \cdot \vec{0} = \vec{0}$
c) $(-1) \cdot \vec{u} = -\vec{u}$
d) If $k \cdot \vec{u} = \vec{0}$ then $k = 0$ or $\vec{u} = \vec{0}$.

# Vector subspace

Let $(V, +, \cdot)$ be a vector space.

Definition: A subset $W$ of $V$ is called a *subspace* if $W$ is itself a vector space with the operations $+$ and $\cdot$ defined on $V$.

Theorem: Let $W$ be a nonempty subset of $V$. Then $W$ is a subspace of $V$ if and only if it is *closed* under addition and scalar multiplication, in other words, if:

i.) For any $\vec{u}, \vec{v}$ in $W$, we have $\vec{u} + \vec{v}$ is in $W$.

ii.) For any scalar $k$ and any vector $\vec{u}$ in $W$ we have $k \cdot \vec{u}$ is in $W$.

# Linear combinations and the span of a set of vectors

Let $(V, +, \cdot)$ be a vector space.

Definition: Let $\vec{v}$ be a vector in $V$. We say that $\vec{v}$ is a *linear combination* of the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}$ if there are scalars $k_1, k_2, \ldots, k_r$ such that

$$\vec{v} = k_1 \cdot \vec{v_1} + k_2 \cdot \vec{v_2} + \ldots + k_r \cdot \vec{v_r}$$

The scalars $k_1, k_2, \ldots, k_r$ are called the *coefficients* of the linear combination.

Definition: Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a set of vectors in $V$.
Let $W$ be the set of all linear combinations of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}$:

$$W = \{k_1 \cdot \vec{v_1} + k_2 \cdot \vec{v_2} + \ldots + k_r \cdot \vec{v_r} \ : \ \text{for all choices of scalars } k_1, k_2, \ldots, k_r\}$$

Then $W$ is called the *span* of the set $S$.
We write:

$$W = \text{ span } S \quad \text{ or } \quad W = \text{ span } \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$$

# Linear combinations and the span of a set of vectors

Let $(V, +, \cdot)$ be a vector space and let $S = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_r}\}$ be a set of vectors in $V$.

$$\text{span}S = \{k_1 \cdot \vec{v_1} + k_2 \cdot \vec{v_2} + \ldots + k_r \cdot \vec{v_r} \; : \; \text{for all scalars } k_1, k_2, \dots, k_r\}$$

(all linear combinations of $\vec{v_1}, \vec{v_2}, \dots, \vec{v_r}$).

Theorem: span $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_r}\}$ is a subspace of $V$.
It is in fact the *smallest* subspace of $V$ that contains all vectors $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_r}\}$.

# Linear independence in a vector space $V$

Let $V$ be a vector space and let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a set of vectors in $V$.

Definition: The set $S$ is called *linearly independent* if the vector equation

$$(*) \quad c_1 \cdot \vec{v_1} + c_2 \cdot \vec{v_2} + \ldots + c_r \cdot \vec{v_r} = \vec{0}$$

has only one solution, the trivial one:

$$c_1 = 0, \; c_2 = 0, \; \ldots, c_r = 0$$

The set is called linearly dependent otherwise, if equation (*) has other solutions besides the trivial one.

Theorem: The set of vectors $S$ is linearly independent if and only if no vector in the set is a linear combination of the other vectors in the set.

The set of vectors $S$ is linearly dependent if and only if one of the vectors in the set is a linear combination of the other vectors in the set.

# The solutions to a homogeneous system of equations $A \cdot \vec{x} = \vec{0}$

Consider a *homogeneous* system of $m$ equations with $n$ unknowns.

In other words,

- let $A$ be an $m \times n$ matrix
- let $\vec{x}$ be an $n \times 1$ matrix (or vector) whose entries are the unknowns $x_1, x_2, \ldots, x_n$
- let $\vec{0}$ denote the $n \times 1$ matrix (vector) whose entries are all 0.

The system can then be written as

$$A \cdot \vec{x} = \vec{0}$$

Theorem: The set of solutions to a *homogeneous* system of $m$ equations with $n$ unknowns is a subspace of $\mathbb{R}^n$.

# Reminder from MA1201 on systems of equations

The case when $\boxed{\# \text{ of equations} = \# \text{ of unknowns}}$

Theorem: Let $A$ be a square matrix in $\mathbf{M}_{n\,n}$.
The following statements are equivalent:

1. $A$ is invertible
2. $\det(A) \neq 0$
3. The homogeneous system $A \cdot \vec{x} = \vec{0}$ has only the trivial solution
4. The system of equations $A \cdot \vec{x} = \vec{b}$ has exactly one solution for every vector $\vec{b}$ in $\mathbb{R}^n$.

The case when $\boxed{\# \text{ of unknowns} > \# \text{ of equations}}$

Theorem: Let $A$ be a matrix in $\mathbf{M}_{m\,n}$, where $n > m$.
Then the homogeneous system $A \cdot \vec{x} = \vec{0}$ has infinitely many solutions.

# Basis in a vector space $V$

Definition: A set $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ of vectors is called a *basis* for $V$ if

1. $S$ is linearly independent
2. $\text{span}\,(S) = V$.

In other words, the set $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ is a basis for $V$ if

1. The equation $c_1 \cdot \vec{v_1} + c_2 \cdot \vec{v_2} + \ldots + c_r \cdot \vec{v_r} = \vec{0}$ has only the trivial solution.
2. The equation $c_1 \cdot \vec{v_1} + c_2 \cdot \vec{v_2} + \ldots + c_r \cdot \vec{v_r} = \vec{b}$ has a solution for every $\vec{b}$ in $V$.

# Standard bases for the most popular vector spaces

- In $\mathbb{R}^2$: $\{\vec{i}, \vec{j}\}$.
- In $\mathbb{R}^3$: $\{\vec{i}, \vec{j}, \vec{k}\}$.
- In $\mathbb{R}^n$: $\{e_1, e_2, \ldots, e_n\}$.
- In $\mathbf{P}_n$: $1, X, X^2, \ldots, X^n$.
- In $\mathbf{M}_{2\,2}$: all matrices with all entries 0 except for one entry, which is 1. There are 4 such matrices.
- In $\mathbf{M}_{m\,n}$: all matrices with all entries 0 except for one entry, which is 1. There are $m \cdot n$ such matrices.

# Dimension of a vector space

Some vector spaces do not have a *finite* basis.
A vector space has many different bases. However,

Theorem: All bases of a finite dimensional vector space have the same number of elements.

Definition: Let $V$ be a finite dimensional vector space. We call *dimension* of $V$ is the number of elements of a basis for $V$.
We use the notation $\dim(V)$ for the dimension of $V$.

Example: Counting the elements of the standard basis of each of the popular vectors spaces, we have that:

- $\dim(\mathbb{R}^n) = n$
- $\dim(\mathbf{P}_n) = n + 1$
- $\dim(\mathbf{M}_{m\,n} = m \cdot n$

# The null space, row space and column space of a matrix

Definition: Given an $m \times n$ matrix $A$, we define:

1. The *column space* of $A =$ the subspace of $\mathbb{R}^m$ spanned by the columns of $A$. Its dimension is called the rank $(A)$.
2. The *row space* of $A =$ the subspace of $\mathbb{R}^n$ spanned by its rows.
3. The *null space* of $A =$ the solution space of $A \cdot \vec{x} = \vec{0}$.
   It is a subspace of $\mathbb{R}^n$. Its dimension is called the nullity $(A)$.

Theorem: Let $A$ be an $m \times n$ matrix. Then

(a) rank $(A)+$ nullity $(A) = n$

(b) dim column space of $A =$ dim row space of $A$
$=$ rank $(A) =$ number of leading 1s in the RREF of $A$.

# Finding bases for the null space, row space and column space of a matrix

Given an $m \times n$ matrix $A$

1. Reduce the matrix $A$ to the *reduced row echelon form* $R$.
2. Solve the system $R \cdot \vec{x} = \vec{0}$. Find a basis for the solutions space.
   The same basis for the solution space of $R \cdot \vec{x} = \vec{0}$ is a basis for the null space of $A$.
3. Consider the non-zero rows of $R$. They form a basis for the row space of $R$.
   The same basis for the row space of $R$ is a basis for the row space of $A$.
4. Take the columns of $R$ with leading 1s. They form a basis for the column space of $R$.
   The corresponding column vectors in $A$ form a basis for the column space of $A$.

# Gaussian elimination: reduced row echelon form (RREF)

Example: $A = \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$ is in RREF.

Definition: To be in RREF, a matrix $A$ must satisfy the following 4 properties (if it satisfies just the first 3 properties, it is in REF):

1. If a row does not have only zeros, then its first nonzero number is a 1. We call this a *leading* 1.
2. The rows that contain only zeros (if there are any) are at the bottom of the matrix.
3. In any consecutive rows that do not have only zeros, the leading 1 in the lower row is farther right than the leading 1 in the row above.
4. Each column that contains a leading 1 has zeros everywhere else in that column.