The definition of a vector space \((V, +, \cdot)\)

1. For any \(\vec{u}\) and \(\vec{v}\) in \(V\), \(\vec{u} + \vec{v}\) is also in \(V\).
2. For any \(\vec{u}\) and \(\vec{v}\) in \(V\), \(\vec{u} + \vec{v} = \vec{v} + \vec{u}\).
3. For any \(\vec{u}\), \(\vec{v}\), \(\vec{w}\) in \(V\), \(\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}\).
4. There is an element in \(V\) called the zero or null vector, which we denote by \(\vec{0}\), such that for all \(\vec{u}\) in \(V\) we have \(\vec{0} + \vec{u} = \vec{u}\).
5. For every \(\vec{u}\) in \(V\), there is a vector called the negative of \(\vec{u}\) and denoted \(-\vec{u}\), such that \(-\vec{u} + \vec{u} = \vec{0}\).
6. If \(k\) is any scalar in \(\mathbb{R}\) and \(\vec{u}\) is any vector in \(V\), then \(k \cdot \vec{u}\) is a vector in \(V\).
7. For any scalar \(k\) in \(\mathbb{R}\) and any vectors \(\vec{u}\) and \(\vec{v}\) in \(V\), 
   \[k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}\]
8. For any scalars \(k\) and \(m\) in \(\mathbb{R}\) and any vector \(\vec{u}\) in \(V\), 
   \[(k + m) \cdot \vec{u} = k \cdot \vec{u} + m \cdot \vec{u}\]
9. For any scalars \(k\) and \(m\) in \(\mathbb{R}\) and any vector \(\vec{u}\) in \(V\), 
   \[k \cdot (m \cdot \vec{u}) = (k m) \cdot \vec{u}\]
10. For any vector \(\vec{u}\) in \(V\), \(1 \cdot \vec{u} = \vec{u}\).
What determines a vector space?

- A nonempty set $V$ whose elements are called vectors.
- An operation $+$ called vectors addition, such that

  \[ \text{vector} + \text{vector} = \text{vector} \]

  In other words, we have closure under vector addition.
- An operation $\cdot$ called scalar multiplication, such that

  \[ \text{scalar} \cdot \text{vector} = \text{vector} \]

  In other words, we have closure under scalar multiplication.
- The remaining 8 axioms are all satisfied.
Some basic identities in a vector space

**Theorem:** Let $V$ be a vector space. The following statements are always true.

a) $0 \cdot \vec{u} = \vec{0}$

b) $k \cdot \vec{0} = \vec{0}$

c) $(-1) \cdot \vec{u} = -\vec{u}$

d) If $k \cdot \vec{u} = \vec{0}$ then $k = 0$ or $\vec{u} = \vec{0}$. 
Let \((V, +, \cdot)\) be a vector space.

**Definition**: A subset \(W\) of \(V\) is called a *subspace* if \(W\) is itself a vector space with the operations \(+\) and \(\cdot\) defined on \(V\).

**Theorem**: Let \(W\) be a nonempty subset of \(V\). Then \(W\) is a subspace of \(V\) if and only if it is *closed* under addition and scalar multiplication, in other words, if:

i.) For any \(\vec{u}, \vec{v}\) in \(W\), we have \(\vec{u} + \vec{v}\) is in \(W\).

ii.) For any scalar \(k\) and any vector \(\vec{u}\) in \(W\) we have \(k \cdot \vec{u}\) is in \(W\).
Linear combinations and the span of a set of vectors

Let \((V, +, \cdot)\) be a vector space.

**Definition:** Let \(\vec{v}\) be a vector in \(V\). We say that \(\vec{v}\) is a *linear combination* of the vectors \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\) if there are scalars \(k_1, k_2, \ldots, k_r\) such that

\[
\vec{v} = k_1 \cdot \vec{v}_1 + k_2 \cdot \vec{v}_2 + \ldots + k_r \cdot \vec{v}_r
\]

The scalars \(k_1, k_2, \ldots, k_r\) are called the *coefficients* of the linear combination.

**Definition:** Let \(S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}\) be a set of vectors in \(V\). Let \(W\) be the set of all linear combinations of \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\):

\[
W = \{k_1 \cdot \vec{v}_1 + k_2 \cdot \vec{v}_2 + \ldots + k_r \cdot \vec{v}_r : \text{for all choices of scalars } k_1, k_2, \ldots, k_r\}
\]

Then \(W\) is called the *span* of the set \(S\).

We write:

\[
W = \text{span } S \quad \text{or} \quad W = \text{span } \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}
\]
Linear combinations and the span of a set of vectors

Let \((V, +, \cdot)\) be a vector space and let \(S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}\) be a set of vectors in \(V\).

\[
\text{span}S = \{k_1 \cdot \vec{v}_1 + k_2 \cdot \vec{v}_2 + \ldots + k_r \cdot \vec{v}_r : \text{ for all scalars } k_1, k_2, \ldots, k_r\}
\]
(all linear combinations of \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\)).

**Theorem:** \(\text{span} \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}\) is a subspace of \(V\).
It is in fact the *smallest* subspace of \(V\) that contains all vectors \(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}\).
Linear independence in a vector space $V$

Let $V$ be a vector space and let $S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \}$ be a set of vectors in $V$.

**Definition:** The set $S$ is called *linearly independent* if the vector equation

\[ c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \ldots + c_r \cdot \vec{v}_r = \vec{0} \]

has only one solution, the trivial one:

\[ c_1 = 0, \ c_2 = 0, \ldots, c_r = 0 \]

The set is called linearly dependent otherwise, if equation (*) has other solutions besides the trivial one.

**Theorem:** The set of vectors $S$ is linearly independent if and only if no vector in the set is a linear combination of the other vectors in the set.

The set of vectors $S$ is linearly dependent if and only if one of the vectors in the set is a linear combination of the other vectors in the set.
The solutions to a homogeneous system of equations \( A \cdot \vec{x} = \vec{0} \)

Consider a *homogeneous* system of \( m \) equations with \( n \) unknowns.

In other words,
- let \( A \) be an \( m \times n \) matrix
- let \( \vec{x} \) be an \( n \times 1 \) matrix (or vector) whose entries are the unknowns \( x_1, x_2, \ldots, x_n \)
- let \( \vec{0} \) denote the \( n \times 1 \) matrix (vector) whose entries are all 0.

The system can then be written as

\[
A \cdot \vec{x} = \vec{0}
\]

**Theorem:** The set of solutions to a *homogeneous* system of \( m \) equations with \( n \) unknowns is a subspace of \( \mathbb{R}^n \).
Reminder from MA1201 on systems of equations

The case when # of equations = # of unknowns

**Theorem:** Let $A$ be a square matrix in $M_{n,n}$.
The following statements are equivalent:

1. $A$ is invertible
2. $\det(A) \neq 0$
3. The homogeneous system $A \cdot \vec{x} = \vec{0}$ has only the trivial solution
4. The system of equations $A \cdot \vec{x} = \vec{b}$ has exactly one solution for every vector $\vec{b}$ in $\mathbb{R}^n$.

The case when # of unknowns > # of equations

**Theorem:** Let $A$ be a matrix in $M_{m,n}$, where $n > m$.
Then the homogeneous system $A \cdot \vec{x} = \vec{0}$ has infinitely many solutions.
**Basis in a vector space** $V$

**Definition**: A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ of vectors is called a *basis* for $V$ if

1. $S$ is linearly independent
2. span $(S) = V$.

In other words, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a basis for $V$ if

1. The equation $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_r \mathbf{v}_r = \mathbf{0}$ has only the trivial solution.
2. The equation $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_r \mathbf{v}_r = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $V$. 
Standard bases for the most popular vector spaces

- In $\mathbb{R}^2$: $\{\vec{i}, \vec{j}\}$.
- In $\mathbb{R}^3$: $\{\vec{i}, \vec{j}, \vec{k}\}$.
- In $\mathbb{R}^n$: $\{e_1, e_2, \ldots, e_n\}$.
- In $\mathbb{P}_n$: $1, X, X^2, \ldots, X^n$.
- In $\mathbb{M}_{2 \times 2}$: all matrices with all entries 0 except for one entry, which is 1. There are 4 such matrices.
- In $\mathbb{M}_{m \times n}$: all matrices with all entries 0 except for one entry, which is 1. There are $m \cdot n$ such matrices.
Dimension of a vector space

Some vector spaces do not have a *finite* basis. A vector space has many different bases. However,

**Theorem:** All bases of a finite dimensional vector space have the same number of elements.

**Definition:** Let $V$ be a finite dimensional vector space. We call *dimension* of $V$ is the number of elements of a basis for $V$. We use the notation $\dim(V)$ for the dimension of $V$.

**Example:** Counting the elements of the standard basis of each of the popular vectors spaces, we have that:

- $\dim(\mathbb{R}^n) = n$
- $\dim(P_n) = n + 1$
- $\dim(M_{m,n}) = m \cdot n$
The null space, row space and column space of a matrix

Definition: Given an $m \times n$ matrix $A$, we define:

1. The *column space* of $A$ = the subspace of $\mathbb{R}^m$ spanned by the columns of $A$. Its dimension is called the rank ($A$).
2. The *row space* of $A$ = the subspace of $\mathbb{R}^n$ spanned by its rows.
3. The *null space* of $A$ = the solution space of $A \cdot \vec{x} = \vec{0}$.
   It is a subspace of $\mathbb{R}^n$. Its dimension is called the nullity ($A$).

Theorem: Let $A$ be an $m \times n$ matrix. Then

(a) \( \text{rank } (A) + \text{nullity } (A) = n \)
(b) \( \text{dim column space of } A = \text{dim row space of } A = \text{rank } (A) = \text{number of leading 1s in the RREF of } A. \)
Finding bases for the null space, row space and column space of a matrix

Given an $m \times n$ matrix $A$

1. Reduce the matrix $A$ to the *reduced row echelon form* $R$.

2. Solve the system $R \cdot \vec{x} = \vec{0}$. Find a basis for the solutions space. The *same* basis for the solution space of $R \cdot \vec{x} = \vec{0}$ is a basis for the null space of $A$.

3. Consider the non-zero rows of $R$. They form a basis for the row space of $R$. The *same* basis for the row space of $R$ is a basis for the row space of $A$.

4. Take the columns of $R$ with leading 1s. They form a basis for the column space of $R$. The *corresponding* column vectors in $A$ form a basis for the column space of $A$. 
Gaussian elimination: reduced row echelon form (RREF)

Example:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

is in RREF.

Definition: To be in RREF, a matrix \( A \) must satisfy the following 4 properties (if it satisfies just the first 3 properties, it is in REF):

1. If a row does not have only zeros, then its first nonzero number is a 1. We call this a \textit{leading} 1.
2. The rows that contain only zeros (if there are any) are at the bottom of the matrix.
3. In any consecutive rows that do not have only zeros, the leading 1 in the lower row is farther right than the leading 1 in the row above.
4. Each column that contains a leading 1 has zeros everywhere else in that column.