General linear transformations

Definition: Let V, W be two vector spaces. A function $T\colon V \to W$ is called a *linear transformation* from V to W if the following hold for all vectors u, v in V and for all scalars k.

(i) $T(u+v) = T(u) + T(v)$ (additivity)

(ii) $T(ku) = kT(u)$ (homogeneity)

If V and W are the same, we call a linear transformation from V to V a *linear operator*.

Theorem: A function $T\colon V \to W$ is a linear transformation if and only if for all vectors v_1, v_2 in V and for all scalars k_1, k_2 we have

$$T(k_1\, v_1 + k_2\, v_2) = k_1\, T(v_1) + k_2\, T(v_2)$$

General linear transformations

Theorem (basic properties of linear transformations): If T is a linear transformation then

a) $T(\vec{0}) = \vec{0}$

b) $T(-v) = -T(v)$

c) $T(u - v) = T(u) - T(v)$

Theorem: If $T\colon V \to W$ is a linear transformation, $S = \{v_1, v_2, \ldots, v_n\}$ is a basis in V, then for any vector v in V we can evaluate $T(v)$ by

$$T(v) = c_1\, T(v_1) + c_2\, T(v_2) + \ldots + c_n\, T(v_n)$$

where $v = c_1\, v_1 + c_2\, v_2 + \ldots + c_n\, v_n$.

Kernel and range of a linear transformation

Definition: Let $T\colon V \to W$ is a linear transformation.

- The set of all vectors v in V for which $T(v) = \vec{0}$ is called the *kernel* of T.
  We denote the kernel of T by $\ker(T)$.
- The set of all outputs (images) $T(v)$ of vectors in V via the transformation T is called the *range* of T.
  We denote the range of T by $R(T)$.

Theorem: If $T\colon V \to W$ is a linear transformation, then $\ker(T)$ is a *subspace* of V, while $R(T)$ is a subspace of W.

Definition: If V and W are *finite* dimensional vector spaces and $T\colon V \to W$ is a linear transformation, then we call

- $\dim \ker(T) =$ nullity of T
- $\dim R(T) =$ rank of T

Theorem: If V and W are finite dimensional vector spaces and $T\colon V \to W$ is a linear transformation, then

$$\text{rank } (T) + \text{ nullity } (T) = \dim(V)$$

One-to-one and onto functions

Definition (one-to-one function): A function $f\colon X \to Y$ is called *one-to-one* if to distinct inputs it assigns distinct outputs.
More precisely, f is 1-1 means: if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. This is logically equivalent to saying that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Definition (onto function): A function $f\colon X \to Y$ is called *onto* if every element in Y is an output of f.
More precisely, f is onto if for every y in Y there is at least one x in X such that $f(x) = y$.

Linear transformations are functions, so being one-to-one or onto applies (makes sense) for them as well.

Isomorphism

Theorem: A linear transformation $T\colon V \to W$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

Theorem: Let $T\colon V \to V$ be a linear operator, where V is a finite dimensional vector space.
The following statements are equivalent.

a) T is one-to-one
b) $\ker(T) = \{\vec{0}\}$
c) T is onto.

Definition: A linear transformation $T\colon V \to W$ which is one-to-one and onto is called an *isomorphism*.
Two vector spaces V and W are called *isomorphic* if there is an isomorphism $T\colon V \to W$.

Examples: P_{n-1} is isomorphic to $\mathbb{R}^n$. $M_{2\times 2}(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$.

Theorem: Every n dimensional vector space is isomorphic to $\mathbb{R}^n$.

The matrix for a linear transformation: definition

We have:

- An n-dimensional vector space V with a basis $B = \{u_1, u_2, \ldots, u_n\}$.
- An m-dimensional vector space W with a basis B'.
- A linear transformation $T\colon V \to W$.

Definition: The *matrix for T relative to the bases B and B'* is the $m \times n$ matrix $[T]_{B',B}$ defined by

$$[T]_{B',B} = \big[\, [T(u_1)]_{B'} \,|\, [T(u_2)]_{B'} \,|\, \ldots \,|\, [T(u_n)]_{B'} \big]$$

Relative to these bases, we can think of the linear transformation T as simply the multiplication transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by the matrix $[T]_{B',B}$. More precisely, we have the following relation:

$$\big[T(x)\big]_{B'} = [T]_{B',B} \cdot [x]_B$$

The matrix for a linear transformation: properties

In class we have learned the following.

Theorem: If $T: V \to V$ is a linear operator and if B is a basis for V, then the following are equivalent:

(a) T is one-to-one.

(b) $[T]_{B,B}$ is invertible.

Moreover, if these conditions hold, then

$$[T^{-1}]_{B,B} = [T]_{B,B}^{-1}$$

But in fact much more is true: given any linear transformation T from a vector space V to another vector space W, the matrix for T in two chosen basis $[T]_{B',B}$ completely encodes whether the transformation is one-to-one, onto or an isomorphism.

The matrix for a linear transformation: more properties

Let V be an n-dimensional vector space with a basis B and let W be an m-dimensonal vector space with a basis B'. Let $T\colon V \to W$ be a linear transformation.

Theorem: The following are equivalent:

(a) T is one-to-one.

(b) The null space of $[T]_{B,B}$ is $\{\vec{0}\}$.

(c) $\operatorname{nullity}[T]_{B',B} = 0$.

Theorem: The following are equivalent:

(a) T is onto.

(b) The column space of $[T]_{B,B}$ is $\mathbb{R}^m$.

(c) $\operatorname{rank}[T]_{B',B} = \dim(W)$.

Theorem: The following are equivalent:

(a) T is an isomorphism

(b) $\dim(V) = \dim(W)$ and $[T]_{B',B}$ is invertible.