Inner products

Definition: An *inner product* on a real vector space V is an operation (function) that assigns to each pair of vectors $(\vec{u}, \vec{v})$ in V a scalar $\langle \vec{u}, \vec{v} \rangle$ satisfying the following axioms:

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ (Symmetry)
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ (Additivity)
3. $\langle k\, \vec{u}, \vec{v} \rangle = k\, \langle \vec{u}, \vec{v} \rangle$ (Homogeneity)
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$ (Positivity)

Theorem (basic properties): Given vectors $\vec{u}, \vec{v}, \vec{w}$ in an inner product space V, and a scalar k, the following properties hold:

- $\langle \vec{\mathrm{o}}, \vec{v} \rangle = \langle \vec{v}, \vec{\mathrm{o}} \rangle = 0$
- $\langle \vec{u} - \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle$
- $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- $\langle \vec{u}, \vec{v} - \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{w} \rangle$
- $\langle \vec{u}, k\vec{v} \rangle = k\, \langle \vec{u}, \vec{v} \rangle$

Norm and distance in an inner product space

Definition: If V is a real inner product space then we define

- The norm (or length) of $\vec{v}$:

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

- The distance between $\vec{u}$ and $\vec{v}$:

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u} - \vec{v}, \vec{u} - \vec{v} \rangle}$$

Theorem (basic properties): Given vectors $\vec{u}, \vec{v}$ in an inner product space V, and a scalar k, the following properties hold:

- $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ iff $\vec{v} = \vec{0}$.
- $\|k\vec{v}\| = |k| \, \|\vec{v}\|$
- $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$
- $d(\vec{u}, \vec{v}) \geq 0$ and $d(\vec{u}, \vec{v}) = 0$ iff $\vec{u} = \vec{v}$.

Angle between vectors

Theorem (Cauchy-Schwarz): If u and v are vectors in an inner vector space, then

$$\left|\langle u, v\rangle\right| \leq \|u\| \, \|v\|$$

Definition: The angle between two vectors u and v in an inner vector space is defined as

$$\theta = \cos^{-1} \frac{\langle u, v\rangle}{\|u\| \, \|v\|}$$

Theorem (the triangle inequality): If u, v and w are vectors in an inner vector space, then

- $\|u + v\| \leq \|u\| + \|v\|$
- $d(u, v) \leq d(u, w) + d(w, v)$

Orthogonality

Definition: Two vectors u and v in an inner vector space are called *orthogonal* if $\langle u, v \rangle = 0$.

Clearly $u \perp v$ iff the angle between them is $\theta = \frac{\pi}{2}$.

Theorem (the Pythagorean theorem): If u and v are orthogonal vectors in an inner vector space, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Definition: Let W be a subspace of an inner product space V. The set of vectors in V which are orthogonal to every vector in W is called the *orthogonal complement* of V and it is denoted by $W^\perp$.

Theorem: The orthogonal complement has the following properties:

- $W^\perp$ is a subspace of V.
- $W \cap W^\perp = \{\vec{0}\}$.
- If V has finite dimension then $(W^\perp)^\perp = W$.