Eigenvalues and eigenvectors of an $n \times n$ matrix A

Definition: A vector $\vec{x}$ in $\mathbb{R}^n$ is called an *eigenvector* of the matrix A if $\vec{x} \neq \vec{0}$ and $A \cdot \vec{x}$ is a scalar multiple of $\vec{x}$, that is, if there is a scalar λ called an *eigenvalue* such that

$$A \cdot \vec{x} = \lambda \cdot \vec{x} \tag{1}$$

The equation above is called the *eigenvalue equation*.
Given the matrix A, a scalar solution λ is an eigenvalue of A while the corresponding vector solution x is called the eigenvector of A corresponding to the eigenvalue λ.

The goal is to describe a general procedure for finding eigenvalues and eigenvectors of a matrix A.

Theorem: λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0$$

Definition: If A is an $n \times n$ matrix, the expression $\det(\lambda I - A)$ defines a polynomial of degree n in λ, called the *characteristic polynomial* of A and denoted by $p_A(\lambda)$.

Eigenvalues and eigenvectors of an $n \times n$ matrix A

An immediate consequence of the previous theorem above is that if A is an upper triangular (or a lower triangular, or a diagonal) matrix, then its eigenvalues are exactly the entries on the (main) diagonal.

The following theorem is just a reformulation of the definition of an eigenvector. We use it to find the eigenvectors of a matrix.

Theorem: A nonzero vector $\vec{x}$ is an eigenvector of the matrix A with corresponding eigenvalue λ if and only if $\vec{x}$ is in the *null space* of $\lambda I - A$.

Definition: The null space of $\lambda I - A$ is called the *eigenspace* of A corresponding to the eigenvalue λ. Its dimension is called the *geometric multiplicity* of the eigenvalue λ.

Note that we always have

$$\text{geometric multiplicity of } \lambda \ \leq \ \text{algebraic multiplicity of } \lambda$$

where by algebraic multiplicity we mean the multiplicity of λ as a solution of the algebraic equation $p_A(\lambda) = 0$.

Diagonalization of an $n \times n$ matrix A

Definition: A square matrix A is *diagonalizable* if it is similar to a diagonal matrix, in other words, if there exists an invertible matrix P such that $P^{-1} A P$ is diagonal. In this case we say that P diagonalizes A.

Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

The proof of this theorem gives us a procedure to diagonalize a matrix.

1. Find all eigenvalues and corresponding bases for the eigenspaces. Merge all these bases into a set S.
   If S has *fewer* than n elements, then A is *not* diagonalizable.
   If S has n elements, then A is diagonalizable.
2. Let $P = [\vec{p_1} \;\; \vec{p_2} \;\; \ldots \;\; \vec{p_n}]$ be the matrix whose columns are the vectors in S. Then P diagonalizes A.
3. $P^{-1} A P$ is a diagonal matrix and its diagonal entries are exactly the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to the eigenvectors $\vec{p_1}, \vec{p_2}, \ldots, \vec{p_n}$ (in this order).

Diagonalization of an $n \times n$ matrix A

We have some special cases in which it is relatively easy to determine if a matrix is diagonalizable. It is all due to the following.

Theorem: If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are eigenvectors of A corresponding to distinct eigenvalues, then $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly *independent*.

The following statements then hold.

- If an $n \times n$ matrix A has n eigenvalues that are all *distinct*, then A is diagonalizable.
- If A is upper (or lower) triangular, and the entries on the diagonal are all distinct, then A is diagonalizable.

Application: computing powers of a *diagonalizable* matrix

Let A be an $n \times n$ matrix that happens to be diagonalizable.

1. Diagonalize A.
   That is, find P invertible and D diagonal such that

$$P^{-1}\,A\,P = D$$

2. Solve for A and obtain $A = P\,D\,P^{-1}$.
   From here conclude that

$$A^k = P\,D^k\,P^{-1}$$

3. Compute D^k by simply raising the diagonal entries of D to the k-th power.
   Compute P^{-1}.
   Substitute everything into the formula above and obtain A^k.