Eigenvalues and eigenvectors of an $n \times n$ matrix $A$

**Definition:** A vector $\vec{x}$ in $\mathbb{R}^n$ is called an *eigenvector* of the matrix $A$ if $\vec{x} \neq \vec{0}$ and $A \cdot \vec{x}$ is a scalar multiple of $\vec{x}$, that is, if there is a scalar $\lambda$ called an *eigenvalue* such that

$$A \cdot \vec{x} = \lambda \cdot \vec{x} \quad (1)$$

The equation above is called the *eigenvalue equation*. Given the matrix $A$, a scalar solution $\lambda$ is an eigenvalue of $A$ while the corresponding vector solution $x$ is called the eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

The goal is to describe a general procedure for finding eigenvalues and eigenvectors of a matrix $A$.

**Theorem:** $\lambda$ is an eigenvalue of $A$ if and only if

$$\det(\lambda I - A) = 0$$

**Definition:** If $A$ is an $n \times n$ matrix, the expression $\det(\lambda I - A)$ defines a polynomial of degree $n$ in $\lambda$, called the *characteristic polynomial* of $A$ and denoted by $p_A(\lambda)$.
Eigenvalues and eigenvectors of an $n \times n$ matrix $A$

An immediate consequence of the previous theorem above is that if $A$ is an upper triangular (or a lower triangular, or a diagonal) matrix, then its eigenvalues are exactly the entries on the (main) diagonal.

The following theorem is just a reformulation of the definition of an eigenvector. We use it to find the eigenvectors of a matrix.

**Theorem:** A nonzero vector $\vec{x}$ is an eigenvector of the matrix $A$ with corresponding eigenvalue $\lambda$ if and only if $\vec{x}$ is in the null space of $\lambda I - A$.

**Definition:** The null space of $\lambda I - A$ is called the eigenspace of $A$ corresponding to the eigenvalue $\lambda$. Its dimension is called the geometric multiplicity of the eigenvalue $\lambda$.

Note that we always have

$$\text{geometric multiplicity of } \lambda \leq \text{algebraic multiplicity of } \lambda$$

where by algebraic multiplicity we mean the multiplicity of $\lambda$ as a solution of the algebraic equation $p_A(\lambda) = 0$. 
Diagonalization of an $n \times n$ matrix $A$

**Definition:** A square matrix $A$ is *diagonalizable* if it is similar to a diagonal matrix, in other words, if there exists an invertible matrix $P$ such that $P^{-1} A P$ is diagonal. In this case we say that $P$ diagonalizes $A$.

**Theorem:** An $n \times n$ matrix $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

The proof of this theorem gives us a procedure to diagonalize a matrix.

1. Find all eigenvalues and corresponding bases for the eigenspaces. Merge all these bases into a set $S$. If $S$ has *fewer* than $n$ elements, then $A$ is *not* diagonalizable. If $S$ has $n$ elements, then $A$ is diagonalizable.

2. Let $P = [\vec{p}_1 \; \vec{p}_2 \; \ldots \; \vec{p}_n]$ be the matrix whose columns are the vectors in $S$. Then $P$ diagonalizes $A$.

3. $P^{-1} A P$ is a diagonal matrix and its diagonal entries are exactly the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to the eigenvectors $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ (in this order).
Diagonalization of an $n \times n$ matrix $A$

We have some special cases in which it is relatively easy to determine if a matrix is diagonalizable. It is all due to the following.

**Theorem:** If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors of $A$ corresponding to **distinct** eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly *independent*.

The following statements then hold.

- If an $n \times n$ matrix $A$ has $n$ eigenvalues that are all *distinct*, then $A$ is diagonalizable.
- If $A$ is upper (or lower) triangular, and the entries on the diagonal are all distinct, then $A$ is diagonalizable.
Application: computing powers of a diagonalizable matrix

Let $A$ be an $n \times n$ matrix that happens to be diagonalizable.

1. Diagonalize $A$.
   That is, find $P$ invertible and $D$ diagonal such that
   $$P^{-1} A P = D$$

2. Solve for $A$ and obtain $A = P D P^{-1}$.
   From here conclude that
   $$A^k = P D^k P^{-1}$$

3. Compute $D^k$ by simply raising the diagonal entries of $D$ to the $k$-th power.
   Compute $P^{-1}$.
   Substitute everything into the formula above and obtain $A^k$. 