MA1201 Linear Algebra and Geometry

Exercise set 12

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Glossary

Engelsk	Norsk
conic section	kjeglesnitt
hyperbola	hyperbel
parabola	parabel
ellipse	ellipse
quadratic form	kvadratisk form

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 7.3 - Quadratic forms

Exercise 1 Do exercise 1 in chapter 7.3 of Elementary Linear Algebra.

(a)

$$3x_1^2 + 7x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b)

$$4x_1^2 - 9x_2^2 - 6x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c)

$$9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Exercise 2 Do exercise 11 and 12 in chapter 7.3 of Elementary Linear Algebra.

In both 11 and 12, (a) is an ellipse, (b) is a hyperbola, (c) is a parabola, and (d) is a circle.

Exercise 3 Do exercise 15 in chapter 7.3 of Elementary Linear Algebra.

We are given the equation $11x^2 + 24xy + 4y^2 - 15 = 0$. We can rewrite this as an equation involving a quadratic forms by

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 11 & 12 \\ 12 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 15$$

The eigenvalues of this matrix are roots of $(\lambda - 11)(\lambda - 4) - 12^2 = \lambda^2 - 15\lambda - 100$, thus they equal 20 and -5. By row reducing $(\lambda I - A)$ we find the eigenvectors $\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$. Normalizing these we get the change of basis matrix

$$P = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

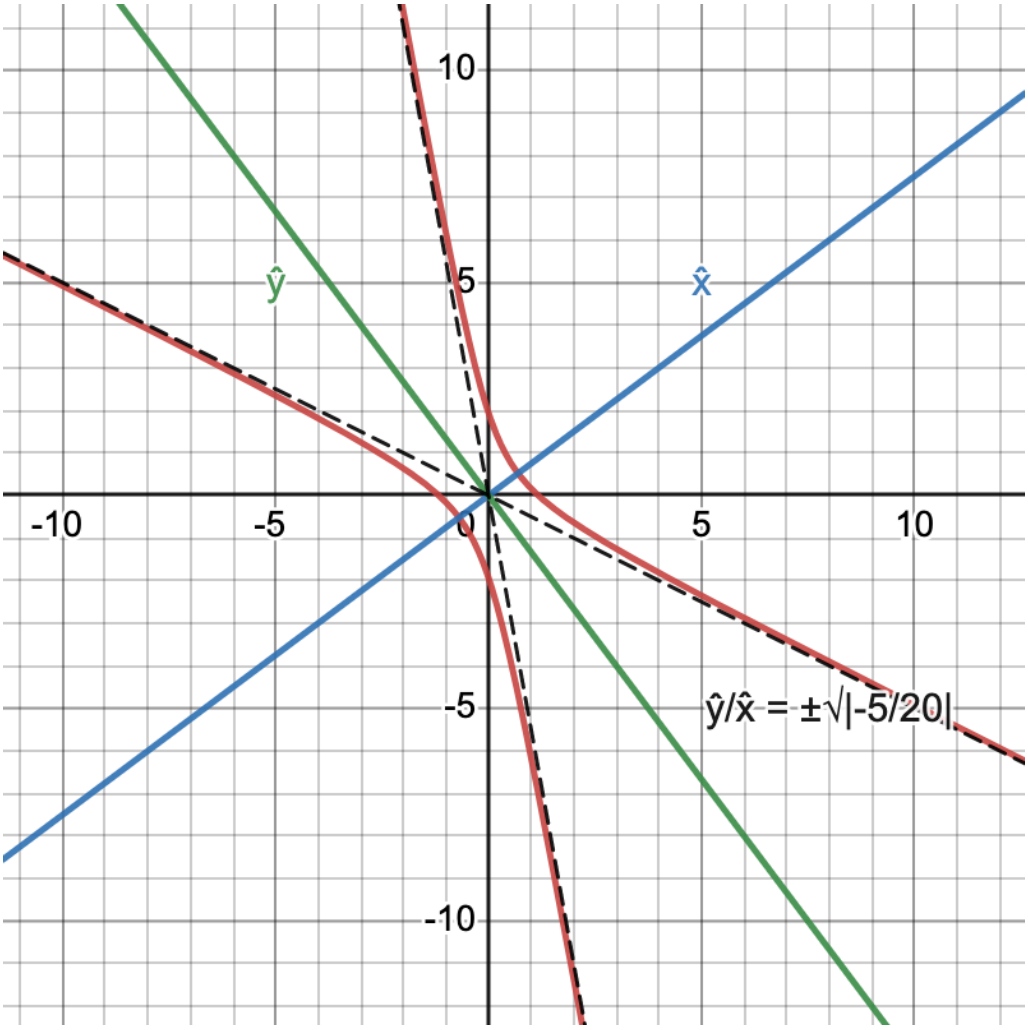
Changing coordinates to $\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = P^T \begin{bmatrix} x \\ y \end{bmatrix}$ we get $20\hat{x}^2 - 5\hat{y}^2 = 15$, which we recognize as a hyperbola since it is a difference of squares.

The angle of rotation is the angle θ such that

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In this case, this will be $\theta = \arccos(4/5) \approx 0.64 \approx 37°$.

A plot of the curve is shown in red below



Here we also give the solution to exercise 16

We are given the equation $x^2+xy+y^2 = 1/2$. We can rewrite this as an equation involving a quadratic form by

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1/2$$

The eigenvalues of this matrix are roots of $(\lambda-1)(\lambda-1)-1/4 = \lambda^2-2\lambda+3/4$, thus they equal $1/2$ and $3/2$. By row reducing $(\lambda I - A)$ we find the eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Normalizing these we get the change of basis matrix

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Since P has determinant -1 it is a reflection. We can make it into a rotation by multiplying one of the columns by -1.

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

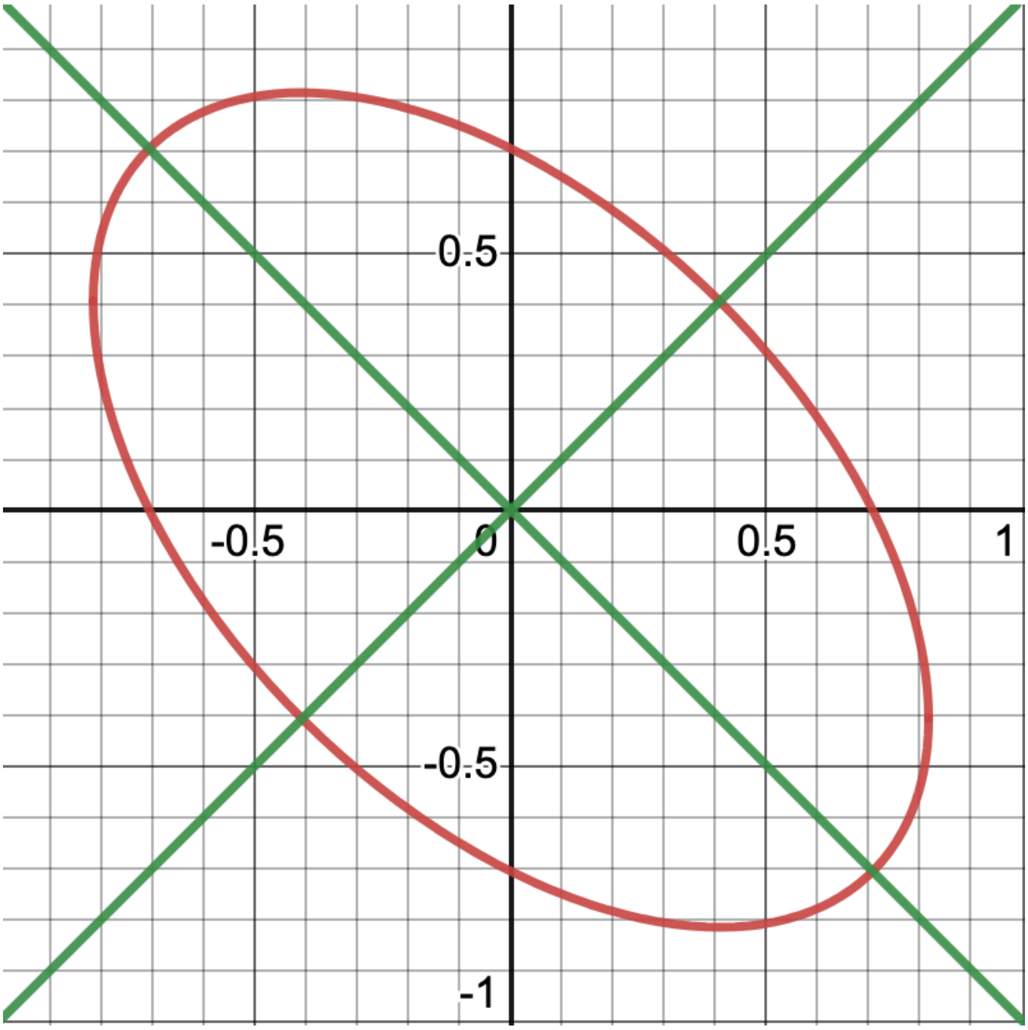
Changing coordinates to $\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = P^T \begin{bmatrix} x \\ y \end{bmatrix}$ we get $\frac{3}{2}\hat{x}^2 + \frac{1}{2}\hat{y}^2 = \frac{1}{2}$, which we recognize as an ellipse since it is a sum of squares.

The angle of rotation is the angle θ such that

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In this case, this will be $\theta = \arctan(-1) = -\frac{\pi}{2} = -45°$.

A plot of the curve is shown in red below



Appendix B - Complex numbers

Exercise 4 Find all the complex solutions to $z^4 = -16$. Write them both in rectangular and polar form, and draw them in the complex plane.

We have $-16 = 16e^{i\pi}$, so if $z = re^{i\theta}$, then $z^4 = r^4 e^{i4\theta}$. This means that $r = \sqrt[4]{16} = 2$ and that $4\theta = \pi + 2n\pi$ for some integer n. So $\theta = \frac{1}{4}\pi + \frac{n}{2}\pi$, and if we restrict θ to $[0, 2\pi)$ the possible solutions are $\frac{1}{4}\pi$, $\frac{3}{4}\pi$, $\frac{5}{4}\pi$ and $\frac{7}{4}\pi$. Thus the solutions are

$$\begin{aligned}
z_1 &= 2e^{i\pi/4} &&= 2(\cos(\pi/4) + i\sin(\pi/4)) &&= \sqrt{2} + i\sqrt{2} \\
z_2 &= 2e^{i3\pi/4} &&= 2(\cos(3\pi/4) + i\sin(3\pi/4)) &&= -\sqrt{2} + i\sqrt{2} \\
z_3 &= 2e^{i5\pi/4} &&= 2(\cos(5\pi/4) + i\sin(5\pi/4)) &&= -\sqrt{2} - i\sqrt{2} \\
z_4 &= 2e^{i7\pi/4} &&= 2(\cos(7\pi/4) + i\sin(7\pi/4)) &&= \sqrt{2} - i\sqrt{2}
\end{aligned}$$