Norwegian University of Science
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Department of Mathematical
Sciences

MA1201 Linear Algebra and Geometry

Exercise set 11

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 7.1 - Orthogonal matrices

Exercise 1 Do exercise 1 in chapter 7.1 of Elementary Linear Algebra.

A square matrix is orthognoal if $A^T A = I$, in which case $A^{-1} = A^T$.

(a)

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus the matrix is orthogonal and is its own inverse.

(b)

$$A^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus the matrix is orthogonal with inverse $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

Exercise 2 Do exercise 26 in chapter 7.1 of Elementary Linear Algebra.

Let us consider a matrix

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Then A is orthognal if and only if $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\}$ forms an orthonormal basis for $\mathbb{R}^2$. In particular we must have $a^2 + b^2 = 1$. This means that (a, b) is a point on the unit circle, hence there is a θ in $[0, 2\pi)$ such that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

We can further say that $\begin{bmatrix} c \\ d \end{bmatrix}$ must be orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$, hence it must equal $s \begin{bmatrix} -b \\ a \end{bmatrix}$ for some real number s. Since it should also have length 1, we must have $|s| = 1$. This leaves us with two options for A, namely

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Both of these give us orthogonal matrices, and so any orthogonal matrix will be of this form.

Chapter 7.2 - Orthogonal diagonalization

Exercise 3 Do exercise 1 and 3 in chapter 7.2 of Elementary Linear Algebra.

Since the matrices are symmetric and every symmetric matrix is diagonalizable we know that the dimension of an eigenspace equals the multiplicity of the eigenvalue in the characteristic polynomial.

(1) The characteristic polynomial is $\det(\lambda I - A) = (\lambda - 1)(\lambda - 4) - 2 \cdot 2 = \lambda^2 - 5\lambda$, which has roots 0 and 5. Each of this has multiplicity 1, so the eigenspaces are 1-dimensional.

(3) The characteristic polynomial is $\det(\lambda I - A) = \lambda^3 - 3\lambda^2$, which has roots 3 and 0. The multiplicity of 0 is 2, so the assoicated eigenspace is 2-dimensional, while the eigenspace associated to 3 is 1-dimensional.

Exercise 4 Find an orthogonal matrix P that orthogonally diagonalizes A and verify that $P^T AP$ is diagonal, where A is the matrix in exercise 1 in chapter 7.2 of Elementary Linear Algebra.

We have seen that the eigenvalues of A are 0 and 5. We find the eigenspaces by solving $(\lambda I - A)\mathbf{x} = \mathbf{0}$:

$$\lambda = 0: \quad \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\lambda = 5: \quad \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

So we have that $(-2, 1)$ is an eigenvector for the value 0, and $(\frac{1}{2}, 1)$ is an eigenvector for the value 5. To get an orthognal matrix from these, we normalize the eigenvectors an use them as columns:

$$P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

We verify that

$$P^T AP = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

Exercise 5 Do exercise 19 and 20 in chapter 7.2 of Elementary Linear Algebra.

For a symmetric matrix we know that eigenvectors that live in distinct eigenspaces are orthogonal. So in exercise 20 we can conclude that no such matrix exists because $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal.

In exercies 19, they are orthognal, and so if we normalize we get an orthonormal basis of eigenvectors. If we let P be the matrix with these as columns and D be the diagonal matrix with the eigenvalues on the diagonal, then $A = PDP^T$ will be a symmetrix matrix with the correct eigenvectors and values.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \end{aligned}$$