Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1201 Linear Algebra and Geometry

Exercise set 10

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 4.7 - Change of basis

Exercise 1 Do exercise 1 in chapter 4.7 of Elementary Linear Algebra.

(a) We set up the augmented matrix and reduce:

$$\begin{bmatrix} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{13}{10} & \frac{-1}{2} \\ 0 & 1 & \frac{-2}{5} & 0 \end{bmatrix}$$

So the transition matrix from B' to B is

$$\begin{bmatrix} \frac{13}{10} & \frac{-1}{2} \\ \frac{-2}{5} & 0 \end{bmatrix}$$

(b) The transition matrix from B to B' will be the inverse of the one we just computed, thus equal to

$$\frac{1}{-1/5} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{13}{10} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-5}{2} \\ -2 & \frac{-13}{2} \end{bmatrix} \tag{1}$$

(c) We set up the augmented matrix and reduce:

$$\begin{bmatrix} 2 & 4 & 3 \\ 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{17}{10} \\ 0 & 1 & \frac{8}{5} \end{bmatrix}$$

$$[\mathbf{w}]_B = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}$$

To compute $[\mathbf{w}]_{B'}$ we multply by the transition matrix

$$[\mathbf{w}]_{B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix} \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

(d) We can compute $[\mathbf{w}]_{B'}$ directly by setting up the augmented matrix and reducing:

$$\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$$

Which verifies that our answer was correct.

Exercise 2 Do exercise 5 in chapter 4.7 of Elementary Linear Algebra.

(a) For something to be a basis it must span V and be linearly independent. We first show linear independence:

Assume $a\mathbf{g}_1 + b\mathbf{g}_2 = 0$. That means $2a\sin x + a\cos x + 3b\cos x = 0$. If we set $x = \frac{\pi}{2}$ then we get $2a = 0$, which means $a = 0$. Then if we set $x = 0$ we get $3b = 0$ which means $b = 0$. So $a = b = 0$ is the only solution, and they are linearly independent.

Since they are two linearly independence vectors, their span is 2-dimensional. And since V is spanned by 2 vectors it must be at most 2-dimensional. Thus $\mathbf{g}_1$ and $\mathbf{g}_2$ span all of V.

Since they are linearly independent and span V, they form a basis.

(b) The transition matrix has columns given by the coordinate vectors

$$P_{B'\to B} = \left[[\mathbf{g}_1]_B[\mathbf{g}_2]_B\right] = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

(c)

$$P_{B\to B'} = P_{B'\to B}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

(d)

$$[\mathbf{h}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \qquad [\mathbf{h}]_{B'} = P_{B\to B'}[\mathbf{h}]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(e) We want to solve $\mathbf{h} = a\mathbf{g}_1 + b\mathbf{g}_2$. We can set up this as a system of equations and rowreduce:

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

This verifies that our answer is correct.

Chapter 5.2 - Diagonalization

Exercise 3 Do exercise 8 in chapter 5.2 of Elementary Linear Algebra.

We begin by finding the eigenvalues of A. The characteristic polynomial is $\det(\lambda I - A) = (\lambda-1)((\lambda-1)^2-1)$ which has roots 0, 1 and 2. Then we find bases for the eigenspaces:

$$\lambda = 0\colon \qquad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenspace has a basis given by the vector $(0, -1, 1)$

$$\lambda = 1\colon \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the eigenspace has a basis given by the vector $(1, 0, 0)$

$$\lambda = 2\colon \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So the eigenspace has a basis given by the vector $(0, 1, 1)$.

Then a choice of P is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

We compute P^{-1} by row reduction:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Performing the matrix multiplication we see that

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Exercise 4 In example 6 on page 307 it is shown that if $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$. Use this to compute A^5, where A is the matrix from the previous exercise.

So $A^5 = PD^5P^{-1}$, where P is the matrix we found in the previous exercise and

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then we can compute A^5 as

$$\begin{aligned} A^5 = PD^5P^{-1} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & 2^5 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 16 \\ 0 & 16 & 16 \end{bmatrix} \end{aligned}$$

Exercise 5 Do exercise 10 in chapter 5.2 of Elementary Linear Algebra.

(a) The characteristic polynomial of A is $\det(\lambda I - A) = (\lambda-3)(\lambda-2)^2$. So the eigenvalues are 3 and 2.

(b) For $\lambda = 3$ we have that

$$3I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $3I - A$ has rank 2.

For $\lambda = 2$ we have

$$2I - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $2I - A$ also has rank 2.

(c) For A to be diagonalizable we need to find 3 linearly independent eigenvectors. We saw that $\lambda I - A$ has rank 2. Using the Rank-Nullity theorem this means that the null space of $\lambda I - A$ is $3 - 2 = 1$-dimensional. The null space of $\lambda I - A$ is exactly the eigenspace of λ, so we have two 1-dimensional eigenspaces. Therefore we can have at most 2 linearly independent eigenvectors, and A is not diagonalizable.