MA1201 Linear Algebra and Geometry

Exercise set 09

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Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 8.1 - Linear transformations

Exercise 1 Do exercise 3 and 4 in chapter 8.1 of Elementary Linear Algebra.

(3) We have $T(-1 \cdot \mathbf{u}) = \|-\mathbf{u}\| = \|\mathbf{u}\| \neq -1 \cdot T(\mathbf{u})$, so T is not linear.

(4) We have seen earlier in the course that

$$T(\mathbf{u}+\mathbf{w}) = (\mathbf{u}+\mathbf{w}) \times \mathbf{v}_0 = \mathbf{u} \times \mathbf{v}_0 + \mathbf{w} \times \mathbf{v}_0 = T(\mathbf{u}) + T(\mathbf{w})$$

and

$$T(\lambda \mathbf{u}) = (\lambda \mathbf{u}) \times \mathbf{v}_0 = \lambda(\mathbf{u} \times \mathbf{v}_0) = \lambda T(\mathbf{u}).$$

So T is a linear transformation.

We have that $\mathbf{u} \times \mathbf{v}_0 = \mathbf{0}$ iff $\mathbf{u}$ is a multiple of $\mathbf{v}_0$, so the kernel of T is $\mathrm{LinSpan}\{\mathbf{v}_0\}$.

Chapter 5.1 - Eigenvalues and eigenvectors

Exercise 2 Do exercise 2 in chapter 5.1 of Elementary Linear Algebra.

$$A\mathbf{x} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $\mathbf{x}$ is an eigenvector with eigenvalue 4.

Exercise 3 Do exercise 6a in chapter 5.1 of Elementary Linear Algebra.

The charcteristic equation is $\det(\lambda I - A) = 0$, which gives us $\lambda^2 - 4\lambda + 3 = 0$. The solutions to this equation are $\lambda = 3$ and $\lambda = 1$, so the eigenvalues of A are 3 and 1. To find the bases for the eigenspaces we rowreduce $\lambda I - A$:

$$\lambda = 3: \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So the eigenspace associated to 3 consists of all vectors where $x_1 - x_2 = 0$ and x_2 is free. A bais is given by $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

We do a similar calculation for $\lambda = 1$:

$$\lambda = 1\colon \qquad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Which gives us basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Exercise 4 Do exercise 25 in chapter 5.1 of Elementary Linear Algebra.

(a) Since the characteristic polynomial has degree $1 + 2 + 3 = 6$, the matrix is 6×6.

(b) Since 0 is not a root of the characteristic polynomial, A does not have 0 as an eigenvalue. That means that the nullspace $N(A)$ is $\{\mathbf{0}\}$, which for a square matrix is equivalent to being invertible. So A is invertible.

(c) A matrix has one eigenspace for each eigenvalue. From the characteristic polynomial we see that we have 3 eigenvalues, and thus 3 eigenspaces.

Exercise 5 Do exercise 33 in chapter 5.1 of Elementary Linear Algebra.

We have that $\mathbf{x}$ is an eigenvector with eigenvalue λ. That means that $A\mathbf{x} = \lambda\mathbf{x}$. If we multiply both sides by A^{-1} we get:

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\lambda\mathbf{x} \\ \mathbf{x} &= A^{-1}\lambda\mathbf{x} \\ \mathbf{x} &= \lambda A^{-1}\mathbf{x} \\ \frac{1}{\lambda}\mathbf{x} &= A^{-1}\mathbf{x} \end{aligned}$$

This is exactly the statement that $\mathbf{x}$ is an eigenvector of A^{-1} with eigenvalue $1/\lambda$, which is what we wanted to prove. $\square$

Exercise 6 Let A be the matrix in exercise 6a in chapter 5.1, considered earlier in this exercise set. Diagonalize A, i.e. find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Verify your solution by checking that $AP = PD$.

Earlier we found the eigenvalues of A to be 3 and 1 and we found corresponding basisvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This gives us that

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

We verify this by computing AP and PD:

$$AP = \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix} = PD$$