MA1201 Linear Algebra and Geometry

Exercise set 08

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Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 4.8 - Row, Column, and Null space

Exercise 1 Do exercise 9a in chapter 4.8 of Elementary Linear Algebra.

If we row reduce the matrix we get

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

The row space is not changed during row reduction, so we see that a basis for the row space is

$$\{\begin{bmatrix} 1 & 0 & -16 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -19 \end{bmatrix}\}.$$

The solutions to $A\mathbf{x} = \mathbf{0}$ is $x = 16z$, $y = 19z$ with z free. Writing the solution in vector form yields

$$\mathbf{x} = z \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

Which means that a basis for the null space is

$$\left\{ \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \right\}$$

Exercise 2 Using the matrices in exercise 9a and 9b, find a basis for the column space.

For the matrix in 9a, if we row reduce the matrix we get

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$$

Now the column space can change during row reduction, but the relations between columns is preserved. Since the two first columns get pivot elements when row reduced they will form a basis for the column space. Thus a basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix} \right\}$$

For the matrix in 9b, if we row reduce the matrix we get

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus a basis for the column space is

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$$

Exercise 3 Do exercise 11a in chapter 4.8 of Elementary Linear Algebra.

Since the matrix is in row echolon form the non-zero rows are linearly independent and the columns with pivot elements are linearly independent. Thus a basis for the row space is

$$\left\{ \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}$$

and a basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Chapter 4.9 - Rank–Nullity

Exercise 4 Do exercise 9 in chapter 4.9 of Elementary Linear Algebra.

The dimension of the row space is equal to the dimension of the column space which is equal to the rank of the matrix. So we can put that into our table.

The rank-nullity theorem tells us that for an $m \times n$-matrix then $\dim(N(A)) + \dim(C(A)) = n$. So $\dim(N(A)) = n - \dim(C(A))$. By transposing A we get the similar formula $\dim(N(A^T)) = m - \dim(R(A))$.

The system $A\mathbf{x} = \mathbf{b}$ is consistent exactly when $\mathbf{b}$ is in the column space of A, this means that A and $[A|\mathbf{b}]$ has the same column space and thus the same rank. Conversely, if $\mathbf{b}$ is not in the column space of A, then the column space of $[A|\mathbf{b}]$ must be bigger, and thus the rank must also be bigger. So the system is inconsistent when the rank of $[A|\mathbf{b}]$ is bigger than the rank of A.

The number of free variables is equal to the dimension of the null space. So the number of paramters of a general solution will be the dimension of the null space, assuming a solution exists.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank($[A\|\mathbf{b}]$)	3	3	1	2	3	0	2
dim($C(A)$)	3	2	1	2	2	0	2
dim($R(A)$)	3	2	1	2	2	0	2
dim($N(A)$)	0	1	2	7	7	4	0
dim($N(A^T)$)	0	1	2	3	3	4	4
Consistent?	Yes	No	Yes	Yes	No	Yes	Yes
#Paramters	0	–	2	7	–	4	0

Exercise 5 Do exercise 30 in chapter 4.9 of Elementary Linear Algebra.

The nullity is the dimension of the solution space $A\mathbf{x} = \mathbf{0}$. Since this has only the trivial solution the nullity is 0. The rank is the number of columns minus the nullity, in this case 6.

Exercise 6 Do exercise 34 in chapter 4.9 of Elementary Linear Algebra.

Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then both matrices have rank 1, but

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So A^2 has rank 1, while B^2 has rank 0.