Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1201 Linear Algebra and Geometry

Exercise set 07

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 4.3 - Spanning sets

Exercise 1 Do exercise 2 in chapter 4.3 of Elementary Linear Algebra.

In order to express the vectors as linear combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, we have to solve the equation $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{b}$ for $\mathbf{b}$ the vector given in each exercise. To do this we can set up the augmented matrix and row reduce:

$$\begin{bmatrix} 2 & 1 & 3 & -9 & 6 & 0 \\ 1 & -1 & 2 & -7 & 11 & 0 \\ 4 & 3 & 5 & -15 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 4 & 0 \\ 0 & 1 & 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

So we have

$$-2\mathbf{u} + \mathbf{v} - 2\mathbf{w} = \begin{bmatrix} -9 \\ -7 \\ -15 \end{bmatrix}$$

$$4\mathbf{u} - 5\mathbf{v} + \mathbf{w} = \begin{bmatrix} 6 \\ 11 \\ 11 \end{bmatrix}$$

$$0\mathbf{u} + 0\mathbf{v} + 0\mathbf{w} = \mathbf{0}$$

Exercise 2 Do exercise 7a in chapter 4.3 of Elementary Linear Algebra.

To determine if vectors span $\mathbb{R}^n$, one methode is to put the vectors as columns in a matrix, row reduce, and see if we get a pivot in every row. When we do this we get

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since every row has a pivot element, the vectors span $\mathbb{R}^3$.

Exercise 3 Do exercise 23 in chapter 4.3 of Elementary Linear Algebra.

That $\{\mathbf{u}, \mathbf{v}\}$ spans V means that for every $\mathbf{w} \in V$, there are coefficients $a, b \in \mathbb{R}$ such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. But then $\mathbf{w} = (a-b)\mathbf{u} + b(\mathbf{u}+\mathbf{v})$, and thus $\{\mathbf{u}, \mathbf{u}+\mathbf{v}\}$ spans V.

Chapter 4.4 - Linear independence

Exercise 4 Do exercise 2 in chapter 4.4 of Elementary Linear Algebra.

(a) To determine if vectors are linearly independent we can put them as columns in a matrix, row reduce, and observe wether every column has a pivot. If we do this we get

$$\begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

showing that the vectors are linearly independent.

(b) Here we have four vectors, so we will get a matrix with four columns, but only three rows. Since we can't have more pivots than rows, it's impossible for every column to have a pivot. Thus the vectors are not linearly independent.

Exercise 5 Do exercise 24 *or* 25 in chapter 4.4 of Elementary Linear Algebra.

Let T be a nonempty subset of S. Then any linear combination of vectors in T can be considered as a linear combination of vectors in S, by considering the coefficients of vectors not in T to be 0. The defintion of S being linearly independent is that a linear combination of vectors in S is only $\mathbf{0}$ if all coefficients are 0. Since linear combinations of vectors in T is just a special case of linear combinations of vectors in S, the only way to make $\mathbf{0}$ is still to have all coefficients be 0. Hence T is linearly independent.

Chapter 4.5 - Basis

Exercise 6 Do exercise 1 in chapter 4.5 of Elementary Linear Algebra.

If we have n vectors in $\mathbb{R}^n$, then they form a basis if and only if the determinant of the matrix with the vectors as columns is non-zero.

$$\det\left(\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\right) = -3 \neq 0$$

Thus the vectors form a basis for $\mathbb{R}^2$.

Exercise 7 Do exercise 3 in chapter 4.5 of Elementary Linear Algebra.

A basis is defined by two properties: it is linearly independent, and it spans the space. Let us first check wether they are linearly independent. A linear combination is on the form

$$a(x^2+1) + b(x^2-1) + c(2x-1) = (a+b)x^2 + 2cx + (a-b-c)$$

If we set this equal to 0 we get a system of equations

$$\begin{aligned} a + b &= 0 \\ 2c &= 0 \\ a - b - c &= 0 \end{aligned}$$

If we set up the corresponding matrix and row reduce we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since we get the identity matrix we know that there is a unique solution, which must be $a = b = c = 0$. Thus the polynomials are linearly independent.

To show that they span the space we need to show that any polynomial in P_2 is a linear combination of the three. That is, given a polynomial $\alpha x^2 + \beta x + \gamma$ we must show that there exists a, b, c such that

$$a(x^2 + 1) + b(x^2 - 1) + c(2x - 1) = \alpha x^2 + \beta x + \gamma$$

This comes down to solving the following system of equations

$$\begin{aligned} a + b &= \alpha \\ 2c &= \beta \\ a - b - c &= \gamma \end{aligned}$$

We already saw that the corresponding matrix reduces to the identity, which implies that the system has a unique solution. Thus the three polynomials does span P_2, and hence they form a basis.