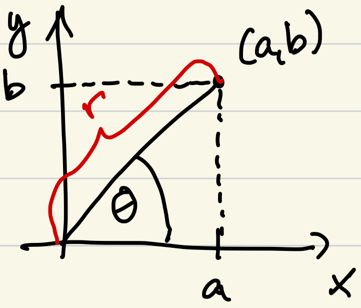
Geometrisk tolkning av multiplikasjon.



Et punkt (a,b) i planet kan også representeres ved polar koordinater $\langle r,\theta\rangle$, der

$r = \sqrt{a^2+b^2}$ – modulusen til $a+bi$

θ = argumentet til $a+bi$.

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \quad \text{og} \quad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

Motsatt, gitt $\langle r,\theta\rangle \longleftrightarrow (a,b) = (r\cos\theta, r\sin\theta)$

$$\Updownarrow$$

$$r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

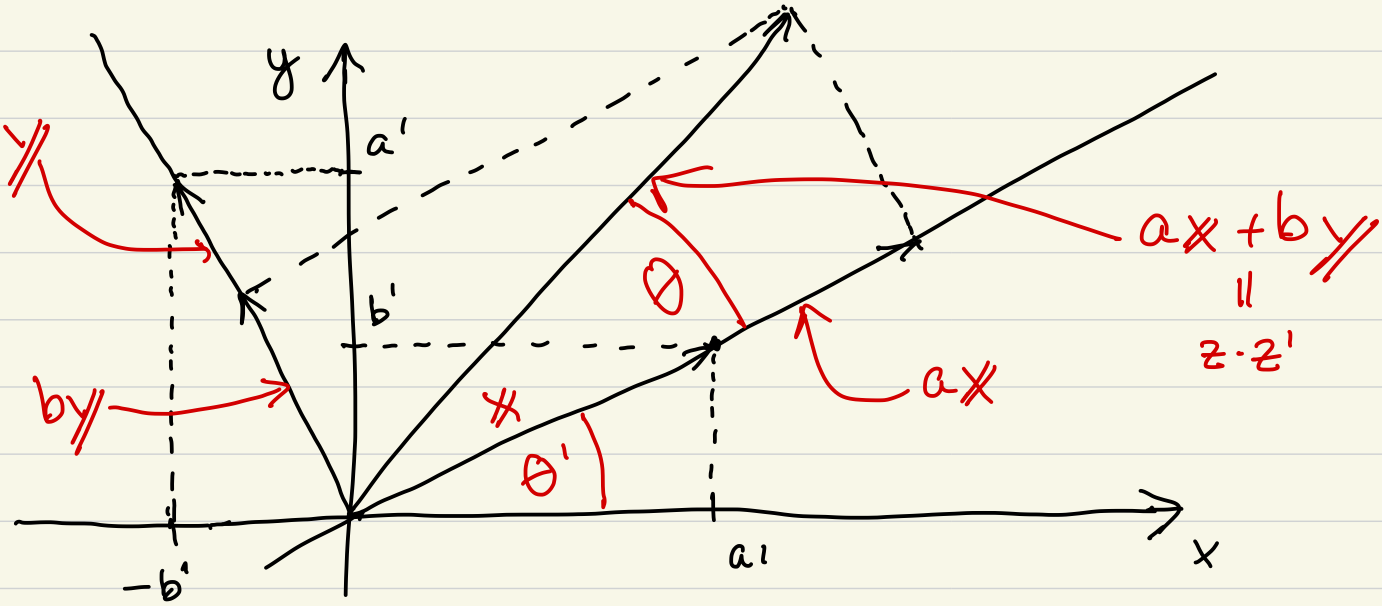
Hvordan tolke multiplikasjonen geometrisk?

(1) Gitt $z = a+bi \longleftrightarrow \langle r,\theta\rangle$

$z' = a'+b'i \longleftrightarrow \langle r',\theta'\rangle$

$$\begin{aligned} z\cdot z' &= (a+bi)(a'+b'i) \\ &= a(a'+b'i) + bi(a'+b'i) \\ &= a(a'+b'i) + b(-b'+a'i) \longleftrightarrow a(\underbrace{a',b'}_{x}) + b(\underbrace{-b',a'}_{y}) \end{aligned}$$

ortogonale vektorer med samme lengde $\|x\| = \|y\|$



Modulusen til $|z\cdot z'| = \sqrt{\|a\mathbf{x}\|^2 + \|b\mathbf{y}\|^2}$

$$= \sqrt{a^2\|\mathbf{x}\|^2 + b^2\|\mathbf{y}\|^2} \qquad \|\mathbf{x}\| = \|\mathbf{y}\|$$

$$= \underset{|z'|}{\underset{\|}{\|\mathbf{x}\|}}\ \underset{|z|}{\underset{\|}{\sqrt{a^2+b^2}}}$$

$$= |z'|\cdot|z|$$

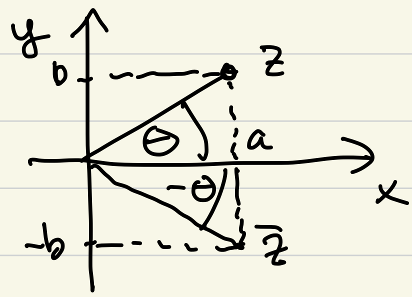
Argumentet θ'' til $z\cdot z'$: Ser at $\theta'' = \theta + \theta'$

Dette gir at

$$z\cdot z' \longleftrightarrow \langle r,\theta\rangle\langle r',\theta'\rangle = \langle rr', \theta+\theta'\rangle.$$

<u>Merk:</u> $z = a + bi \longleftrightarrow \langle r, \theta\rangle$

$\bar{z} = a - bi \longleftrightarrow \langle r, -\theta\rangle$



$$\Rightarrow z^{-1} = \frac{1}{|z|^2}\bar{z} = \frac{1}{r^2}\bar{z} \longleftrightarrow \left\langle \frac{1}{r}, -\theta\right\rangle.$$

② Direkte argument:

$$
\begin{aligned}
z \cdot z' &= r(\cos\theta + i\sin\theta)\, r'(\cos\theta' + i\sin\theta') \\
&= rr'\big[\underbrace{(\cos\theta\cos\theta' - \sin\theta\sin\theta')}\, + i\underbrace{(\cos\theta\sin\theta' + \sin\theta\cos\theta')}\big] \\
&= rr'\big(\cos(\theta+\theta') + i\sin(\theta+\theta')\big)
\end{aligned}
$$

Dette gir

De Moivres formel For alle $n \geq 0$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Bevis: Induksjon på n. Oppgave. $\square$

Eulers formel

$$e^{i\theta} \overset{\text{def}}{=} \cos\theta + i\sin\theta, \qquad \theta \in \mathbb{R}.$$

Hvorfor?

Husk: $e^0 = 1$ og $e^a \cdot e^b = e^{a+b}$, $a, b \in \mathbb{R}$.

- $e^{i0} = \cos 0 + i\sin 0 = 1 + i \cdot 0 = 1.$
- $$\begin{aligned} e^{i(\theta+\theta')} &= \cos(\theta+\theta') + i\sin(\theta+\theta') \\ &= (\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') \\ &= e^{i\theta} \cdot e^{i\theta'} \end{aligned}$$

Merk: $e^{i\theta} = e^{i(\theta + 2n\pi)}$ (cos og sin periodiske med periode 2π)

Proposisjon 7.1

z, z' komplekse tall

(a) $|\bar{z}| = |z|$

(b) $|z \cdot z'| = |z| \cdot |z'|$.

(c) $\left|\frac{1}{z}\right| = \frac{1}{|z|}$, for $z \neq 0$.

(d) $|z + z'| \leq |z| + |z'|$

Bevis: Oppgave. $\square$