Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1201 Linear Algebra and Geometry

Exercise set 12

Glossary

Engelsk	**Norsk**
conic section	kjeglesnitt
hyperbola	hyperbel
parabola	parabel
ellipse	ellipse
quadratic form	kvadratisk form

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 7.1 - Orthogonal matricies

Exercise 1 Do exercise 2 in chapter 7.1 of Elementary Linear Algebra.

(a) The identity matrix is orthogonal and is its own inverse.

(b) We have that

$$\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = 4/5$$

Hence the columns are not orthogonal and so the matrix cannot be orthogonal.

Exercise 2 Do exercise 26 in chapter 7.1 of Elementary Linear Algebra.

Let us consider a matrix

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Then A is orthognal if and only if $\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right\}$ forms an orthonormal basis for $\mathbb{R}^2$. In particular we must have $a^2 + b^2 = 1$. This means that (a, b) is a point on the unit circle, hence there is a θ in $[0, 2\pi)$ such that

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

We can further say that $\begin{bmatrix} c \\ d \end{bmatrix}$ must be orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$, hence it must equal $s\begin{bmatrix} -b \\ a \end{bmatrix}$ for some real number s. Since it should also have length 1, we must have $|s| = 1$. This leaves us with two options for A, namely

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Both of these give us orthogonal matrices, and so any orthogonal matrix will be of this form.

Chapter 7.2 - Orthogonal diagonalization

Exercise 3 Do exercise 12 in chapter 7.2 of Elementary Linear Algebra.

The characteristic polynomial of the matrix is $\lambda^3 - 2\lambda^2$, so the eigenvalues are 2 and 0. Computing the eigenvectors gives us that $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for the eigenspace associated to 2, and $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace associated to 0.

These are already orthogonal, so to get an orthonormal basis of eigenvectors we just need to normalize. This gives us

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 4 Do exercise 19 and 20 in chapter 7.2 of Elementary Linear Algebra.

For a symmetric matrix we know that eigenvectors that live in distinct eigenspaces are orthogonal. So in exercise 20 we can conclude that no such matrix exists because $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal.

In exercies 19, they are orthognal, and so if we normalize we get an orthonormal basis of eigenvectors. If we let P be the matrix with these as columns and D be the diagonal matrix with the eigenvalues on the diagonal, then $A = PDP^T$ will be a symmetrix matrix with the correct eigenvectors and values.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \end{aligned}$$

Chapter 7.3 - Quadratic forms

Exercise 5 Do exercise 11 and 12 in chapter 7.3 of Elementary Linear Algebra.

In both 11 and 12, (a) is an ellipse, (b) is a hyperbola, (c) is a parabola, and (d) is a circle.

Exercise 6 Do exercise 15 in chapter 7.3 of Elementary Linear Algebra.

We are given the equation $11x^2 + 24xy + 4y^2 - 15 = 0$. We can rewrite this as an equation involving a quadratic forms by

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 11 & 12 \\ 12 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 15$$

The eigenvalues of this matrix are roots of $(\lambda - 11)(\lambda - 4) - 12^2 = \lambda^2 - 15\lambda - 100$, thus they equal 20 and -5. By row reducing $(\lambda I - A)$ we find the eigenvectors $\begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$. Normalizing these we get the change of basis matrix

$$P = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$$

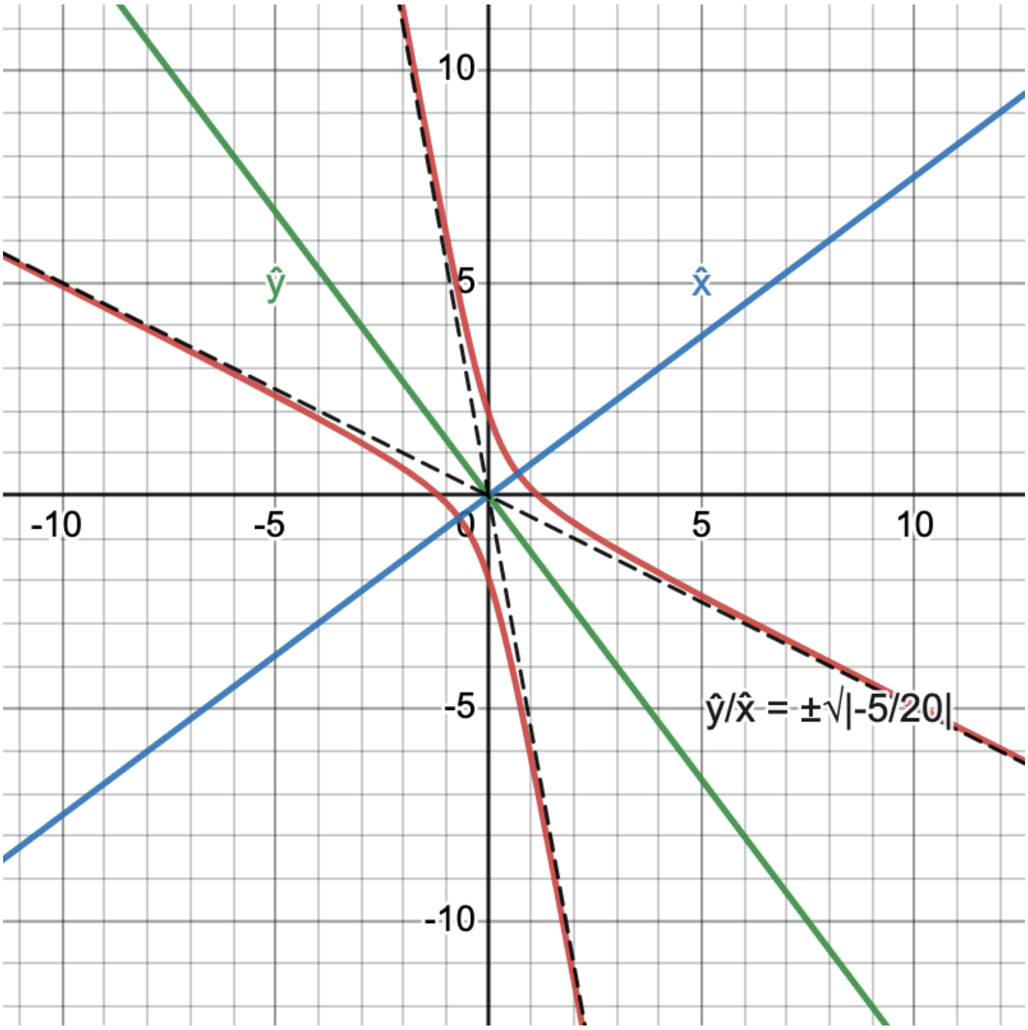
Changing coordinates to $\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = P^T \begin{bmatrix} x \\ y \end{bmatrix}$ we get $20\hat{x}^2 - 5\hat{y}^2 = 15$, which we recognize as a hyperbola since it is a difference of squares.

The angle of rotation is the angle θ such that

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In this case, this will be $\theta = \arccos(4/5) \approx 0.64 \approx 37°$.

A plot of the curve is shown in red below


ŷ
x̂
ŷ/x̂ = ±√|-5/20|
10
5
-5
-10
-10
-5
0
5
10