Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1201 Linear Algebra and Geometry

Exercise set 10

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 5.1 - Eigenvalues and eigenvectors

Exercise 1 Do exercise 1 in chapter 5.1 of Elementary Linear Algebra.

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So $\mathbf{x}$ is an eigenvector with eigenvalue -1.

Exercise 2 Do exercise 5a in chapter 5.1 of Elementary Linear Algebra.

The charcteristic equation is $\det(\lambda I - A) = 0$, which gives us $\lambda^2 - 4\lambda - 5 = 0$. The solutions to this equation are $\lambda = -1$ and $\lambda = 5$, so the eigenvalues of A are -1 and 5. To find the bases for the eigenspaces we rowreduce $\lambda I - A$:

$$\lambda = -1: \quad \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

So the eigenspace associated to -1 consists of all vectors where $x_1 + 2x_2 = 0$ and x_2 is free. A bais is given by $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

We do a similar calculation for $\lambda = 5$:

$$\lambda = 5: \quad \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Which gives us basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Exercise 3 Do exercise 25 in chapter 5.1 of Elementary Linear Algebra.

(a) Since the characteristic polynomial has degree $1 + 2 + 3 = 6$, the matrix is 6×6.

(b) Since 0 is not a root of the characteristic polynomial, A does not have 0 as an eigenvalue. That means that the nullspace $N(A)$ is $\{\mathbf{0}\}$, which for a square matrix is equivalent to being invertible. So A is invertible.

(c) A matrix has one eigenspace for each eigenvalue. From the characteristic polynomial we see that we have 3 eigenvalues, and thus 3 eigenspaces.

Exercise 4 Do exercise 33 in chapter 5.1 of Elementary Linear Algebra.

We have that $\mathbf{x}$ is an eigenvector with eigenvalue λ. That means that $A\mathbf{x} = \lambda\mathbf{x}$. If we multiply both sides by A^{-1} we get:

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\lambda\mathbf{x} \\ \mathbf{x} &= A^{-1}\lambda\mathbf{x} \\ \mathbf{x} &= \lambda A^{-1}\mathbf{x} \\ \frac{1}{\lambda}\mathbf{x} &= A^{-1}\mathbf{x} \end{aligned}$$

This is exactly the statement that $\mathbf{x}$ is an eigenvector of A^{-1} with eigenvalue $1/\lambda$, which is what we wanted to prove. □

Chapter 5.2 - Diagonalization

Exercise 5 Let A be the matrix in exercise 5a in chapter 5.1, considered earlier in this exercise set. Diagonalize A, i.e. find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Verify your solution by checking that $AP = PD$.

Earlier we found the eigenvalues of A to be -1 and 5 and we found corresponding basisvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This gives us that

$$P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

We verify this by computing AP and PD:

$$AP = \begin{bmatrix} 2 & 5 \\ -1 & 5 \end{bmatrix} = PD$$

Exercise 6 In example 6 on page 307 it is shown that if $A = PDP^{-1}$ then $A^k = PD^kP^{-1}$. Use this to compute A^5, where A is the matrix from the previous exercise.

So $A^5 = PD^5P^{-1}$. We first compute P^{-1} to be

$$P^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Then we can compute A^5 as

$$\begin{aligned} A^5 = PD^5P^{-1} &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^5 & 0 \\ 0 & 5^5 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3125 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3125 \\ -1 & 3125 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} 1041 & 2084 \\ 1042 & 2083 \end{bmatrix} \end{aligned}$$

Exercise 7 Do exercise 10 in chapter 5.2 of Elementary Linear Algebra.

(a) The characteristic polynomial of A is $\det(\lambda I - A) = (\lambda - 3)(\lambda - 2)^2$. So the eigenvalues are 3 and 2.

(b) For $\lambda = 3$ we have that

$$3I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $3I - A$ has rank 2.

For $\lambda = 2$ we have

$$2I - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $2I - A$ also has rank 2.

(c) For A to be diagonalizable we need to find 3 linearly independent eigenvectors. We saw that $\lambda I - A$ has rank 2. Using the Rank-Nullity theorem this means that the null space of $\lambda I - A$ is $3 - 2 = 1$-dimensional. The null space of $\lambda I - A$ is exactly the eigenspace of λ, so we have two 1-dimensional eigenspaces. Therefore we can have at most 2 linearly independent eigenvectors, and A is not diagonalizable.