MA1201 Linear Algebra and Geometry

Exercise set 08

Norwegian University of Science and Technology
Department of Mathematical Sciences

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 4.5 - Coordinates

Exercise 1 Do exercise 11 in chapter 4.5 of Elementary Linear Algebra.

(a) We set up the augmented matrix and rowreduce:

$$\begin{bmatrix} 2 & 3 & 1 \\ -4 & 8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{5}{28} \\ 0 & 1 & \frac{3}{14} \end{bmatrix}$$

So $[\mathbf{w}]_S = \begin{bmatrix} \frac{5}{28} \\ \frac{3}{14} \end{bmatrix}$.

(b) We set up the augmented matrix and rowreduce:

$$\begin{bmatrix} 1 & 0 & a \\ 1 & 2 & b \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$$

So $[\mathbf{w}]_S = \begin{bmatrix} a \\ \frac{b-a}{2} \end{bmatrix}$.

Exercise 2 Do exercise 13a in chapter 4.5 of Elementary Linear Algebra.

We set up the augmented matrix and rowreduce:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{v}]_S = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Chapter 4.6 - Dimension

Exercise 3 Do exercise 7 in chapter 4.6 of Elementary Linear Algebra.

(a) $3x - 2y + 5z = 0$ has solutions $x = \frac{2}{3}y - \frac{5}{3}z$, which in vectorform we can write as

$$\begin{bmatrix} \frac{2}{3}y - \frac{5}{3}z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}$$

Therefore the space is 2-dimensional with a basis given by

$$\left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) $x - y = 0$ has solutions $x = y$, which in vectorform we can write as

$$\begin{bmatrix} y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the space is 2-dimensional with a basis given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(c) The line is described as all vectors on the form

$$\begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Therefore the space is 1-dimensional with a basis given by

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}$$

(d) The vectors can be written as

$$\begin{bmatrix} a \\ a + c \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore the space is 2-dimensional with a basis given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Exercise 4 Do exercise 9 in chapter 4.6 of Elementary Linear Algebra (You may assume $n = 3$ if you prefer).

1. Let E_{ij} be the $n \times n$-matrix which is 1 in coordinate (i, j) and 0 elsewhere. Then a basis for the diagonal matrices is $\{E_{ii}\}_{i=1}^{n}$, thus the space is n-dimensional.
2. A basis for the symmetric matrices is $\{E_{ij}+E_{ji}\}_{j \geq i}$ so the space is $\frac{n^2+n}{2}$-dimensional.
3. A basis for the upper triangular matrices is $\{E_{ij}\}_{j \geq i}$ so the space is $\frac{n^2+n}{2}$-dimensional.

Chapter 4.7 - Change of basis

Exercise 5 Do exercise 1 in chapter 4.7 of Elementary Linear Algebra.

(a) We set up the augmented matrix and reduc:

$$\begin{bmatrix} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{13}{10} & \frac{-1}{2} \\ 0 & 1 & \frac{-2}{5} & 0 \end{bmatrix}$$

So the transition matrix from B' to B is

$$\begin{bmatrix} \frac{13}{10} & \frac{-1}{2} \\ \frac{-2}{5} & 0 \end{bmatrix}$$

(b) The transition matrix from B to B' will be the inverse of the one we just computed, thus equal to

$$\frac{1}{-1/5} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{13}{10} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-5}{2} \\ -2 & \frac{-13}{2} \end{bmatrix} \tag{1}$$

(c) We set up the augmented matrix and reduce:

$$\begin{bmatrix} 2 & 4 & 3 \\ 2 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{17}{10} \\ 0 & 1 & \frac{8}{5} \end{bmatrix}$$

$$[\mathbf{w}]_B = \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix}$$

To compute $[\mathbf{w}]_{B'}$ we multply by the transition matrix

$$[\mathbf{w}]_{B'} = \begin{bmatrix} 0 & -\frac{5}{2} \\ -2 & -\frac{13}{2} \end{bmatrix} \begin{bmatrix} -\frac{17}{10} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

(d) We can compute $[\mathbf{w}]_{B'}$ directly by setting up the augmented matrix and reducing:

$$\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -7 \end{bmatrix}$$

Which verifies that our answer was correct.

Exercise 6 Do exercise 5 in chapter 4.7 of Elementary Linear Algebra.

(a) For something to be a basis it must span V and be linearly independent. We first show linear independence:

Assume $a\mathbf{g}_1 + b\mathbf{g}_2 = 0$. That means $2a \sin x + a \cos x + 3b \cos x = 0$. If we set $x = \frac{\pi}{2}$ then we get $2a = 0$, which means $a = 0$. Then if we set $x = 0$ we get $3b = 0$ which means $b = 0$. So $a = b = 0$ is the only solution, and they are linearly independent.

Since they are two linearly independence vectors, their span is 2-dimensional. And since V is spanned by 2 vectors it must be at most 2-dimensional. Thus $\mathbf{g}_1$ and $\mathbf{g}_2$ span all of V.

Since they are linearly independent and span V, they form a basis.

(b) The transition matrix has columns given by the coordinate vectors

$$P_{B' \to B} = \left[[\mathbf{g}_1]_B [\mathbf{g}_2]_B \right] = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

(c)

$$P_{B \to B'} = P_{B' \to B}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

(d)

$$[\mathbf{h}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \qquad [\mathbf{h}]_{B'} = P_{B \to B'} [\mathbf{h}]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(e) We want to solve $\mathbf{h} = a\mathbf{g}_1 + b\mathbf{g}_2$. We can set up this as a system of equations and rowreduce:

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

This verifies that our answer is correct.