Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1201 Linear Algebra and Geometry

Exercise set 05

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 2.1 - determinants and cofactors

Exercise 1 Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find A_{12}, the cofactor at index $(1, 2)$.

To find the cofactor we first remove the first row and second column.

$$\begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{-1} \\ 1 & \cancel{1} & 0 \\ 0 & \cancel{0} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the cofactor is given by $(-1)^{1+1} \cdot (-1)^{2+1} \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = -1$

Exercise 2 Exercise 15 in chapter 2.1 of Elementary Linear Algebra.

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}\right) &= (\lambda - 2)(\lambda + 4) + 5 \\ &= \lambda^2 + 2\lambda - 3 \\ &= (\lambda - 1)(\lambda + 3) \end{aligned}$$

This is 0 when $\lambda = 1$ or $\lambda = -3$.

Exercise 3 Exercise 21 in chapter 2.1 of Elementary Linear Algebra.

If we choose the middle column then the expansion becomes

$$\det\left(\begin{bmatrix} -3 & 0 & 7 \\ 25 & 1 & \\ -1 & 0 & 5 \end{bmatrix}\right) = 5 \cdot ((-3) \cdot 5 - 7 \cdot (-1)) = -40$$

Exercise 4 Exercise 30 in chapter 2.1 of Elementary Linear Algebra.

The determinant of a triangular matrix is the product of teh diagonal elements, so

$$\det\left(\begin{bmatrix}1 & 1 & 1 & 1\\0 & 2 & 2 & 2\\0 & 0 & 3 & 3\\0 & 0 & 0 & 4\end{bmatrix}\right) = 1\cdot 2\cdot 3\cdot 4 = 24$$

Chapter 2.2 - determinants and row reduction

Exercise 5 Exercise 9 in chapter 2.2 of Elementary Linear Algebra.

We row reduce the matrix keeping track of what type of row operations we perform.

$$\begin{bmatrix}3 & -6 & 9\\-2 & 7 & -2\\0 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3\\-2 & 7 & -2\\0 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3\\0 & 3 & 4\\0 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3\\0 & 0 & -11\\0 & 1 & 5\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 3\\0 & 1 & 5\\0 & 0 & -11\end{bmatrix}$$

The determinant of the reduced matrix is $1\cdot 1\cdot(-11) = -11$. The operations we have done that change the determiant is to divide one row by 3, and to swap two rows. Thus the determinant of the original matrix is $(-11)\cdot 3\cdot(-1) = 33$.

Exercise 6 Exercise 15 in chapter 2.2 of Elementary Linear Algebra.

We can obtain this matrix from the original by first swapping the first and second row, then swapping the second and third row. Thus the determiannt has changed by a factor of $(-1)\cdot(-1)$, meaning it hasn't changed at all. The determinant is -6.

Exercise 7 Exercise 22 in chapter 2.2 of Elementary Linear Algebra.

If we take 2 times the first row and add it to the third row we see that we get a row of all zeros. Thus the determinant is 0.

Chapter 2.3 - Cramer's rule

Exercise 8 Calculate the determinant of

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & -1\end{bmatrix}$$

The determiannt is -2.

Exercise 9 Find the adjoint of the matrix above. What's the inverse?

The cofactor matrix is

$$\begin{bmatrix}-1 & 0 & -1\\0 & -2 & 0\\-1 & 0 & 1\end{bmatrix}$$

To find the adjoint we take the transpose of this which gives us the same thing. The inverse is given by

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \begin{bmatrix} frac12 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$