Norwegian University of Science
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Department of Mathematical
Sciences

MA1201 Linear Algebra and Geometry

Exercise set 02

Glossary

Engelsk	Norsk
norm	norm / lengde / størrelse
unit vector	enhetsvektor
acute angle	spiss vinkel
obtuse angle	stump vinkel

Compulsory exercises

Hand in your solutions to these exercises. All answers must be justified.

Chapter 3.1 - Vectors in $\mathbb{R}^n$

Exercise 1 Exercise 12b in chapter 3.1 of Elementary Linear Algebra.

$$3(2\mathbf{u}-\mathbf{v}) = 6\mathbf{u}-3\mathbf{v} = 6\begin{bmatrix}1\\2\\-3\\5\\0\end{bmatrix} - 3\begin{bmatrix}0\\4\\-1\\1\\2\end{bmatrix} = \begin{bmatrix}6\\12\\-18\\30\\0\end{bmatrix} - \begin{bmatrix}0\\12\\-3\\3\\6\end{bmatrix} = \begin{bmatrix}6\\0\\-15\\27\\-6\end{bmatrix}$$

Exercise 2 Exercise 14 in chapter 3.1 of Elementary Linear Algebra.

$$\begin{aligned} 2\mathbf{u} - \mathbf{v} + \mathbf{x} &= 7\mathbf{x} + \mathbf{w} \\ 6\mathbf{x} &= 2\mathbf{u} - \mathbf{v} - \mathbf{w} \\ \mathbf{x} &= \frac{1}{3}\mathbf{u} - \frac{1}{6}\mathbf{v} - \frac{1}{6}\mathbf{w} \\ &= \begin{bmatrix} -5/6 \\ -1/6 \\ -1/6 \\ 11/6 \\ -5/6 \end{bmatrix} \end{aligned}$$

Exercise 3 Exercise 17 in chapter 3.1 of Elementary Linear Algebra.

Looking entrywise at $a\mathbf{u} + b\mathbf{v} = (1, -4, 9, 18)$ gives us the system of equations

$$\begin{aligned} a + 2b &= 1 \\ -a + \ \ b &= -4 \\ 3a \qquad &= 9 \\ 5a - 3b &= 18 \end{aligned}$$

When we set up the augmented matrix and row reduce we get

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & -4 \\ 3 & 0 & 9 \\ 5 & -3 & 18 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which means that the solution is $a = 3$ and $b = -1$.

Chapter 3.2 - Dot products

Exercise 4 Exercise 1a in chapter 3.2 of Elementary Linear Algebra.

We calculate that $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$. Thus $\frac{1}{2\sqrt{3}}\mathbf{v}$ is a unit vector pointing in the same direction as $\mathbf{v}$, and $-\frac{1}{2\sqrt{3}}\mathbf{v}$ is a unit vector pointing in the opposite direction. So the answer is

$$-\frac{1}{2\sqrt{3}}\mathbf{v} = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

Exercise 5 Exercise 3a and 3b in chapter 3.2 of Elementary Linear Algebra.

We have

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(2+1)^2 + (-2-3)^2 + (3+4)^2} = \sqrt{83} \approx 9.1$$

and

$$\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{2^2 + (-2)^2 + 3^2} + \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{17} + \sqrt{26} \approx 9.2.$$

Exercise 6 Exercise 11b in chapter 3.2 of Elementary Linear Algebra.

The distance between **u** and **v** is given by

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(0+3)^2 + (-2-2)^2 + (-1-4)^2 + (1-4)^2} = \sqrt{59}.$$

The cosine of the angle is given by $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. We calculate

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 0 \cdot (-3) + (-2) \cdot 2 + (-1) \cdot 4 + 1 \cdot 4 = -4 \\
\|\mathbf{u}\| &= \sqrt{0^2 + (-2)^2 + (-1)^2 + 1^2} = \sqrt{6} \\
\|\mathbf{v}\| &= \sqrt{(-3)^2 + 2^2 + 4^2 + 4^2} = 3\sqrt{5} \\
\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} &= \frac{-4}{\sqrt{6} \cdot 3\sqrt{5}} = -\frac{4}{3\sqrt{30}}.
\end{aligned}$$

Since this is negative it means that the angle between **u** and **v** is obtuse.

Exercise 7 Exercise 15 in chapter 3.2 of Elementary Linear Algebra.

In (a) **u** is a vector and $(\mathbf{v} \cdot \mathbf{w})$ is a scalar. The dot-product is taken between vectors, so this doesnt make sense.

In (c) $\mathbf{u} \cdot \mathbf{v}$ is a scalar. We don't take the norm of a scalar, so this doesnt make sense.

The remaining two expressions make sense.

Exercise 8 Exercise 17a in chapter 3.2 of Elementary Linear Algebra.

Cauchy-Schwarz says that $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$. We calculate:

$$\begin{aligned}
|\mathbf{u} \cdot \mathbf{v}| &= |-3 \cdot 2 + 1 \cdot (-1) + 0 \cdot 3| = 7 \\
\|\mathbf{u}\| &= \sqrt{(-3)^2 + 1^2 + 0^2} = \sqrt{10} \\
\|\mathbf{v}\| &= \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14} \\
\|\mathbf{u}\|\|\mathbf{v}\| &= \sqrt{140} \approx 11.8
\end{aligned}$$

We see that the inequality holds, since $7 \leq 11.8$.