Eksamen H2016

Oppg 4 (a)

$$M = \left\{ \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = x_2 + 2x_3 \right\}$$

Må vise:

(1) $M \neq \emptyset$

(2) $\bar{x}, \bar{y} \in M \implies \bar{x} + \bar{y} \in M$

(3) $\bar{x} \in M,\ a \in \mathbb{R} \implies a\bar{x} \in M$

(1) $M \neq \emptyset$ siden $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in M$

(2) Anta $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}$ og $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in M$, dvs

$$x_1 = x_2 + 2x_3 \quad \text{og} \quad y_1 = y_2 + 2y_3$$

Har

$$\bar{x} + \bar{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$$

og

$$\begin{aligned} (x_1 + y_1) &= (x_2 + 2x_3) + (y_2 + 2y_3) \\ &= (x_2 + y_2) + 2(x_3 + y_3) \end{aligned}$$

Så $\bar{x} + \bar{y} \in M$

(3) Anta $\bar{x} \in \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in M$ og $a \in \mathbb{R}$. Har $a\bar{x} = \begin{pmatrix} ax_1 \\ ax_2 \\ ax_3 \end{pmatrix}$

$$\begin{aligned} (ax_1) &= a(x_2 + 2x_3) \\ &= (ax_2) + 2\cdot(ax_3) \end{aligned}$$

Så $a\bar{x} \in M$.

Derfor er M et underrom av $\mathbb{R}^3$.

(b)

$$\bar{x} \in M \iff x_1 = x_2 + 2x_3$$
$$\iff x_1 - x_2 - 2x_3 = 0$$
$$\iff (1 \quad -1 \quad -2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$
$$\iff A\bar{x} = 0 \qquad \text{hvor } A = (1 \quad -1 \quad -2)$$

Så M = nullrommet til A! A er allerede på redusert trappeform. Frie variabler:

$$x_2 = s \ , \ x_3 = t$$
$$\implies x_1 = s + 2t$$

$$N(A) = \left\{ \begin{pmatrix} s+2t \\ s \\ t \end{pmatrix} \middle| \ s,t \in \mathbb{R} \right\} = \left\{ s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \middle| \ s,t \in \mathbb{R} \right\}$$

Basis for $N(A) = M$ er $\{\bar{u}_1, \bar{u}_2\}$ hvor

$$\bar{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \bar{u}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Gram-Schmidt på $\{\bar{u}_1, \bar{u}_2\}$:

(1) $\bar{v}_1 = \bar{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(2) $\bar{v}_2 = \bar{u}_2 - \dfrac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \cdot \bar{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \dfrac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Så en ortogonal basis for M er

$$\{\bar{v}_1, \bar{v}_2\}$$

hvor $\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(c)

For $\bar{w} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ er

$$\text{proj}_M \bar{w} = \frac{\bar{w} \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1 + \frac{\bar{w} \cdot \bar{v}_2}{\|\bar{v}_2\|^2} \bar{v}_2$$

$$= \frac{4}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{-3}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$