Norwegian University of Science and Technology
Department of Mathematical Sciences

Hand-in assignment
Deadline: 31.03.2023

Hand-in assignment, 4 topics, 3 pages
Deadline: 31.03.2023, 16:00

Solutions should be clear, written on paper, tablet or by other means, in English or Norwegian, and handed in as a single PDF in **Inspera**.

Choose **two** topics, and write systematically about item (a) and (b) in each of the two topics. The expected extent is approximately four pages in total, but will vary with your way of writing. Express yourself rigorously, explaining to someone who knows mathematics, but not the precise topic.

You may work and hand in individually or in pairs of two, but you should be ready to answer questions individually about the handed-in material. The hand-in itself should be signed with your full name, as well as the name of your possible coworker.

The hand-in assignment is weighted as **30%** of the complete course grade.[1]

- **Topic 1. Functions $\mathbf{r} : I \subseteq \mathbb{R} \to \mathbb{R}^3$**

(a) Describe how functions $\mathbf{r} : I \subseteq \mathbb{R} \to \mathbb{R}^3$ are related with curves in $\mathbb{R}^3$. Mention important concepts like parametrization, arc length, position, velocity $\mathbf{v}$, speed, and acceleration $\mathbf{a}$. Describe also the Frenet-frame and its importance for the fundamental theorem of space curves.

(b) Explain the intuition behind the concepts of curvature and torsion. Provide an example to illustrate your explanation. Deduce also the formulas

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}, \qquad \tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot (\frac{d}{dt}\mathbf{a})}{|\mathbf{v} \times \mathbf{a}|^2}.$$

- **Topic 2. Differentiability of functions $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$**

(a) Describe the basic theory of differentiability for functions $U \subseteq \mathbb{R}^2 \to \mathbb{R}$, including partial ∂_x and directional $\partial_{\mathbf{u}}$ derivatives, gradients ∇, differentiability, and continuous differentiability (C^1). Highlight the differences with respect to functions $\mathbb{R} \to \mathbb{R}$.

[1] Check the general rules for this kind of assignment here. By illness lasting more than one week of the project period, use the standard NTNU form to be eligible for a re-sit exam/project.

(b) Let $\mathbf{r} : I \subseteq \mathbb{R} \to \mathbb{R}^2$ be a parametrization of a curve $\mathcal{C}$ defined by

$$t \mapsto (r_1(t), r_2(t)), \qquad \text{where } r_1, r_2 : I \subseteq \mathbb{R} \to \mathbb{R}.$$

Moreover, let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, and let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be defined by

$$t \mapsto f(\mathbf{r}(t)).$$

Explain under which conditions the following chain rule holds

$$g'(t) = \nabla f(\mathbf{r}(t)) \cdot \frac{d}{dt}\mathbf{r}(t).$$

Use similar chain rules to deduce that

$$\partial_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Be precise in your assumptions and calculations.

- **Topic 3. Extreme values of functions $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$**

(a) Describe the basic theory of extremal values for functions $U \subseteq \mathbb{R}^2 \to \mathbb{R}$, including critical, singular, and boundary points, local and global maxima/minima, saddle points, the second derivative test, and any necessary concepts of sets or functions.

(b) Let $U \subseteq \mathbb{R}^2$ be a compact (bounded and closed) domain, and let $f : U \to \mathbb{R}$ be a $C^1(U; \mathbb{R})$-function. Hence, f has extreme values on U. We are then looking for these extreme values which can be found at the critical points of f inside U, or at the boundary of U. Define the constraint $g : U \to \mathbb{R}$ to be a $C^1(U; \mathbb{R})$-function such that we can reformulate the problem as

$$\text{maximize/minimize } f(x, y) \text{ with respect to } g(x, y) \leq 0.$$

Define a new constraint such that we can use the method of Lagrange multipliers to find the extreme values of f *both* inside U *and* at the boundary of U.

Hint: Usually we can only find the extreme values of f at the boundary of U by using the method of Lagrange multipliers. The question is therefore: How to adapt this method to also find the critical points of f?

- **Topic 4. Heat kernel**

Define $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ by

$$(x, t) \mapsto \frac{1}{(4\pi t)^{\frac{n}{2}}} \mathrm{e}^{-\frac{|x|^2}{4t}},$$

where $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

(a) Explain why f is continuous, why $\partial_t f$ and $\partial_{x_i} f$, for all $i \in \{1, 2, \dots, n\}$, all exist and are continuous, and why $\partial_{x_j}\partial_{x_i} f$, for all $i, j \in \{1, 2, \dots, n\}$, all exist and are continuous. Then show that f satisfies the partial differential equation

$$\partial_t f(x,t) - \Delta f(x,t) = 0 \qquad \text{for all } (x,t) \in \mathbb{R}^n \times (0,\infty),$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}\partial_{x_i} = \sum_{i=1}^n \partial_{x_i}^2$.

(b) Argue that

$$\int_{\mathbb{R}^n} f(x,t)dx = 1.$$

Hint: You might want to reduce the problem to calculating

$$\int_{\mathbb{R}} \mathrm{e}^{-u^2} du.$$