MA1103 Vector Calculus
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Exercise set 9: Solutions

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1 (14.2)

Evaluate the double integrals below by iteration.

a) $\iint_T (x - 3y)\,dA$, where T is the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$.

b) $\iint_R \frac{x}{y} e^y\,dA$, where R is the region $0 \le x \le 1$, $x^2 \le y \le x$.

Solution.

a)

We have

$$\begin{aligned}
\iint_T (x-3y)\,dA &= \int_0^a \left(\int_0^{b(1-\frac{x}{a})} (x-3y)\,dy \right) dx \\
&= \int_0^a \left(xy - \frac{3}{2} y^2 \right) \Bigg|_{y=0}^{y=b(1-\frac{x}{a})} dx \\
&= \int_0^a \left[b\Big(x - \frac{x^2}{a}\Big) - \frac{3}{2} b^2 \Big(1 - \frac{2x}{a} + \frac{x^2}{a^2}\Big) \right] dx \\
&= \left(b\frac{x^2}{2} - \frac{b}{a}\frac{x^3}{3} - \frac{3}{2} b^2 x + \frac{3}{2}\frac{b^2 x^2}{a} - \frac{1}{2}\frac{b^2 x^3}{a^2} \right) \Bigg|_0^a \\
&= \frac{a^2 b}{6} - \frac{ab^2}{2}.
\end{aligned}$$

b)

We have

$$\begin{aligned}\iint_R \frac{x}{y}e^y\,dA =& \int_0^1 \frac{e^y}{y}\,dy \int_y^{\sqrt{y}} x\,dx\\ =&\frac{1}{2}\int_0^1 (1-y)e^y\,dy\\ =&\frac{1}{2}\int_0^1 (1-y)\,d(e^y)\\ =&\frac{1}{2}(1-y)e^y\Big|_0^1 + \frac{1}{2}\int_0^1 e^y\,dy\\ =&-\frac{1}{2}+\frac{1}{2}(e-1) = \frac{e}{2}-1.\end{aligned}$$

2 (14.3)

Evaluate

$$I = \iint_S \frac{1}{x+y}\,dA,$$

where S is the square $0 \le x \le 1$, $0 \le y \le 1$,

a) by direct iteration of the double integral,

b) by using the symmetry of the integrand and the domain to write

$$I = 2\iint_T \frac{1}{x+y}\,dA$$

where T is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Solution.

a)

We get

$$\begin{aligned}I = \iint_S \frac{1}{x+y}\,dA =& \int_0^1 \left(\int_0^1 \frac{1}{x+y}\,dy\right)dx\\ =& \int_0^1 \left(\ln(x+y)\Big|_{y=0}^{y=1}\right)dx\\ =& \lim_{c\to 0+}\Big[(x+1)\ln(x+1) - x\ln(x)\Big]\Big|_c^1\\ =& \lim_{c\to 0+} 2\ln(2) - 0 - (c+1)\ln(c+1) + c\ln(c) = 2\ln(2).\end{aligned}$$

b)

We get

$$\begin{aligned}
I =&2 \iint_T \frac{1}{x+y}\, dA \\
=&2 \lim_{c\to 0+} \int_c^1 \left(\int_0^x \frac{1}{x+y}\, dy \right) dx \\
=&2 \lim_{c\to 0+} \int_c^1 \left(\ln(x+y)\Big|_{y=0}^{y=x} \right) dx \\
=&2 \lim_{c\to 0+} \int_c^1 \Big(\ln(2x) - \ln(x) \Big)\, dx \\
=&2\ln(2) \int_0^1 \, dx = 2\ln(2).
\end{aligned}$$

3 (14.4)

Evaluate the double integrals below.

a) $\iint_D \sqrt{x^2+y^2}\, dA$, where D is the disk $x^2+y^2 \le a^2$, $a > 0$.

b) $\iint_S x\, dA$, where S is the disk segment $x^2+y^2 \le 2$, $x \ge 1$.

Solution.

a)

We have

$$\iint_D \sqrt{x^2+y^2}\, dA = \int_0^{2\pi} d\theta \int_0^a r^2\, dr = \frac{2\pi a^3}{3}.$$

b)

We have

$$\begin{aligned}
\iint_S x\, dA =&2 \int_0^{\frac{\pi}{4}} d\theta \int_{\sec(\theta)}^{\sqrt{2}} r\cos(\theta)\cdot r\, dr \\
=&\frac{2}{3} \int_0^{\frac{\pi}{4}} \cos(\theta)\Big(2\sqrt{2} - \sec^3(\theta)\Big)\, d\theta \\
=&\frac{4\sqrt{2}}{3} \sin(\theta)\Big|_0^{\frac{\pi}{4}} - \frac{2}{3}\tan(\theta)\Big|_0^{\frac{\pi}{4}} \\
=&\frac{4}{3} - \frac{2}{3} = \frac{2}{3}.
\end{aligned}$$

4 (14.4)

Find the volume of the region in the first octant below the paraboloid

$$z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Hint: Use the change of variables $x = au$, $y = bv$.

Solution.

Let E be the region in the first quadrant of the xy-plane bounded by the coordinate axes and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The required volume is

$$V = \iint_E \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dxdy.$$

Let $x = au$, $y = bv$. Then

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = ab\, dudv.$$

Thus,

$$V = ab \iint_Q (1 - u^2 - v^2)\, dudv.$$

The region E corresponds to the quarter disk Q: $u^2 + v^2 \leq 1$, $u, v \geq 0$ in the uv-plane. Now transform to polar coordinates in the uv-plane: $u = r\cos(\theta)$, $v = r\sin(\theta)$.

$$\begin{aligned} V =& ab \int_0^{\frac{\pi}{2}} d\theta \int_0^1 (1 - r^2) r\, dr \\ =& \frac{\pi ab}{2} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 = \frac{\pi ab}{8}. \end{aligned}$$

5 (14.4)

Let T be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Evaluate the integral $\iint_T e^{\frac{y-x}{y+x}}\, dA$

a) by transforming to polar coordinates,

b) by using the transformation $u = y - x$, $v = y + x$.

Solution.

a)

$$\begin{aligned} I =& \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos(\theta)+\sin(\theta)}} e^{\frac{\cos(\theta)-\sin(\theta)}{\sin(\theta)+\cos(\theta)}} \cdot r\, dr \\ =& \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{\frac{\cos(\theta)-\sin(\theta)}{\sin(\theta)+\cos(\theta)}} \frac{1}{(\cos(\theta)+\sin(\theta))^2}\, d\theta. \end{aligned}$$

Let $u = \frac{\cos(\theta)-\sin(\theta)}{\sin(\theta)+\cos(\theta)}$, then $du = -\frac{2}{(\cos(\theta)+\sin(\theta))^2}\, d\theta$. Thus,

$$I = \frac{1}{4}\int_{-1}^{1} e^u \, du = \frac{e - e^{-1}}{4}.$$

b)

If $u = y - x$, $v = y + x$, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2,$$

so that $dA = dxdy = \frac{1}{2}\, dudv$. Also, T corresponds to the triangle T' bounded by $u = -v$, $u = v$, and $v = 1$. Thus,

$$\begin{aligned} I =& \frac{1}{2}\iint_{T'} e^{\frac{u}{v}}\, dudv \\ =& \frac{1}{2}\int_0^1 \left(\int_{-v}^{v} e^{\frac{u}{v}}\, du\right) dv \\ =& \frac{1}{2}\int_0^1 \left(ve^{\frac{u}{v}}\right)\Big|_{-v}^{v} dv \\ =& \frac{1}{2}(e - e^{-1})\int_0^1 v\, dv = \frac{e - e^{-1}}{4}. \end{aligned}$$

6 (14.5)

Evaluate the triple integrals below over the indicated region.

a) $\iiint_R (1 + 2x - 3y)\, dV$, over the box $-a \leq x \leq a$, $-b \leq y \leq b$, $-c \leq z \leq c$.

b) $\iiint_R yz^2 e^{-xyz}\, dV$, over the cube $0 \leq x, y, z \leq 1$.

Solution.

a)

R is symmetric about the coordinate planes and has volume $8abc$. Thus

$$\begin{aligned} &\iiint_R (1 + 2x - 3y)\, dV \\ =& \iiint_R dV + 2\iiint_R x\, dV - 3\iiint_R y\, dV \\ =& \text{volume of } R + 0 - 0 = 8abc. \end{aligned}$$

b)

R is the cube $0 \le x, y, z \le 1$. We have

$$\begin{aligned}
&\iiint_R yz^2 e^{-xyz}\, dV \\
&= \int_0^1 z\left(\int_0^1 (-e^{-xyz})\Big|_{x=0}^{x=1} dy\right) dz \\
&= \int_0^1 z\left(\int_0^1 (1-e^{-yz})\, dy\right) dz \\
&= \int_0^1 z\left(1 + \frac{1}{z}e^{-yz}\Big|_{y=0}^{y=1}\right) dz \\
&= \frac{1}{2} + \int_0^1 (e^{-z}-1)\, dz \\
&= \frac{1}{2} - 1 - e^{-z}\Big|_0^1 = \frac{1}{2} - \frac{1}{e}.
\end{aligned}$$

7 **a)** *Old exam problem.*

A region R in space (på norsk: rommet) is bounded by (på norsk: avgrenset av) the surfaces $z = y^2,\ z = 2 - y,\ x = 0,$ and $x = 2$. Sketch the region R and determine its volume.

b) *Old exam problem.*

Let T denote the body (på norsk: legemet) determined by $x \ge 0\ y \ge 0$, and $0 \le z \le 1 - \sqrt{x^2+y^2}$. Given that the mass density (The mass density of an object is defined as its mass per unit volume) of T at a point $(x, y, z) \in T$ is $\delta(x, y, z) = z(x^2 + y^2)$, determine the mass of T (Mass is the quantity of matter in a physical body).

Hint: According to the definition of mass density and mass, the mass of T is denoted by the triple integral $M(T) = \iiint_T \delta\, dV$.

Solution.

a)

A sketch of the region R is given in Figure 1 below. Let $\lambda(R)$ denote the volume of R. Then $\lambda(R) = \iiint_R dz\, dy\, dx$. From the figure, z runs from the cylinder $z = y^2$ to the plane $z = 2 - y$. x runs from 0 to 2, and the range of y is obtained from the intersections of the surfaces $z = y^2$ and $z = 2 - y$. That is, from the solutions of the quadratic equation $y^2 + y - 2 = 0$. The left-hand side of this equation can be factorized as $(y + 2)(y - 1)$ so the solutions are $y = -2$ and $y = 1$. Thus, y runs from -2 to 1. Hence,

$$\lambda(R) = \int_0^2 \int_{-2}^1 \int_{y^2}^{2-y} dz\, dy\, dx = 2\int_{-2}^1 (2 - y - y^2)\, dy = 9.$$

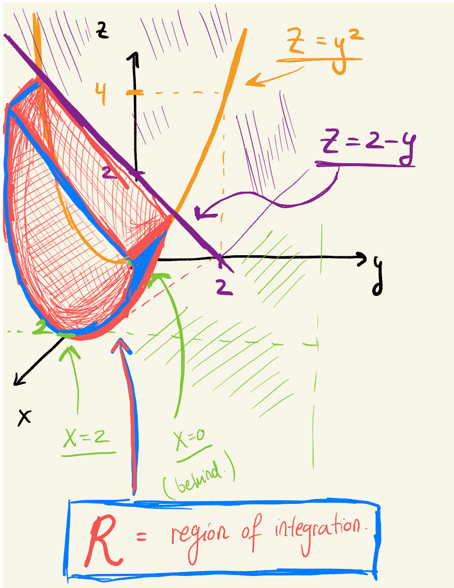

Figure 1: A sketch of the region R in Problem 6 **a)**. It is a parabolic cylinder ($z = y^2$) with symmetry axis along the x-axis (over the interval $[0, 2]$) with a skew plane (the plane $z = 2 - y$) as a "lid".

b)

Let $M(T)$ denote the mass of T. By definition, $M(T) = \iiint_T \delta \, dV$. We use cylindrical coordinates r, θ, z. Then $1 - \sqrt{x^2 + y^2} = 1 - r$. Since $x, y \geq 0$, we find that $\theta \in [0, \frac{\pi}{2}]$. In the (r, z)-plane, the region of integration is given by $0 \leq z \leq 1 - r$. This is a triangle with vertices at $(0, 0)$, $(0, 1)$ and $(1, 0)$. Thus, r runs from 0 to 1. Finally, under this change of variables the density function becomes zr^2, so we find

$$M(T) = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r} zr^3 \, dz \, dr \, d\theta = \frac{\pi}{2} \int_0^1 \frac{1}{2} r^3 (1 - r)^2 \, dr = \frac{\pi}{4} \int_0^1 r^3 (1 - 2r + r^2) \, dr$$
$$= \frac{\pi}{4} \int_0^1 r^3 - 2r^4 + r^5 \, dr = \frac{\pi}{4} \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right) = \frac{\pi}{240}.$$