Norwegian University of Science and Technology
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MA1101 Basic Calculus I
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Exercise set 6: Solutions

1 (12.4)

Let u and v be two twice continuously differentiable real-valued functions defined on an open subset $U \subset \mathbb{R}^2$ which satisfy the so-called *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y),$$
$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$$

for all $(x,y) \in U$. Show that u and v are *harmonic* in U. That is, satisfy the 2-dimensional Laplace equation $\Delta(\cdot) = 0$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, in U.

Hint: it will be necessary to use and refer to a certain theorem.

Solution.

Since u and v are given to be twice continuously differentiable, we have from the theorem on equality of mixed partial derivatives, that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and similarly for u. From $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, differentiating the first equality with respect to x and the second with respect to y, we therefore get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} + \left(-\frac{\partial^2 v}{\partial y \partial x}\right) = 0.$$

This shows that u is harmonic, and the argument for v being harmonic is similar.

2 (12.9)

Find the Taylor series for the given functions near the indicated points.

a) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto 2x^2 - xy - y^2 - 6x - 3y + 5, \qquad (x_0, y_0) = (1, -2)$

b) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto \sin(2x + 3y), \qquad (x_0, y_0) = (0, 0)$

Solution.

a)

We have

$$f(1,-2)=5, \quad f_x(1,-2)=(4x-y-6)\Big|_{(1,-2)}=0,$$
$$f_y(1,-2)=(-x-2y-3)\Big|_{(1,-2)}=0,$$
$$f_{xx}(1,-2)=4, \quad f_{xy}(1,-2)=-1, \quad f_{yy}(1,-2)=-2.$$

This function is a quadratic polynomial, so the third order partial derivative and above are all equal to 0. Thus, the Taylor series for f about $(1,-2)$ is given by

$$\begin{aligned} f(x,y) =& f(1,-2)+(x-1)f_x(1,-2)+(y+2)f_y(1,-2) \\ &+\frac{1}{2!}\Big[(x-1)^2 f_{xx}(1,-2)+2(x-1)(y+2)f_{xy}(1,-2)+(y+2)^2 f_{yy}(1,-2)\Big] \\ =&5+\frac{1}{2}\Big[4(x-1)^2-2(x-1)(y+2)-2(y+2)^2\Big] \\ =&5+2(x-1)^2-(x-1)(y+2)-(y+2)^2. \end{aligned}$$

b)

Recall the Taylor series for $\sin(x)$ at $x=0$ is

$$\sin(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Thus, to our f, we have

$$\begin{aligned} f(x,y)=\sin(2x+3y)&=\sum_{n=0}^{\infty}(-1)^n\frac{(2x+3y)^{2n+1}}{(2n+1)!} \\ =\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}&\sum_{j=0}^{2n+1}\frac{(2n+1)!}{j!(2n+1-j)!}(2x)^j(3y)^{2n+1-j} \\ =\sum_{n=0}^{\infty}\sum_{j=0}^{2n+1}&\frac{(-1)^n 2^j 3^{2n+1-j}}{j!(2n+1-j)!}x^j y^{2n+1-j}, \end{aligned} \tag{1}$$

where we have used the binomial theorem in the second last step that

$$\begin{aligned} &(2x+3y)^{2n+1} \\ =&\binom{2n+1}{0}(2x)^{2n+1}(3y)^0+\binom{2n+1}{1}(2x)^{2n}(3y)^1+\binom{2n+1}{2}(2x)^{2n-1}(3y)^2+\cdots \\ &+\binom{2n+1}{2n}(2x)^1(3y)^{2n}+\binom{2n+1}{2n+1}(2x)^0(3y)^{2n+1} \\ =&\sum_{j=0}^{2n+1}\binom{2n+1}{j}(2x)^{2n+1-j}(3y)^j=\sum_{j=0}^{2n+1}\binom{2n+1}{j}(2x)^j(3y)^{2n+1-j}, \end{aligned}$$

and

$$\binom{2n+1}{j}=\frac{(2n+1)!}{j!(2n+1-j)!}.$$

So (1) is the Taylor series for f about $(0,0)$.

3 (13.1)

Find and classify the critical points of the given functions below.

a) $f(x, y) = x^2 + 2y^2 - 4x + 4y$

b) $f(x, y) = x \sin(y)$

Solution.

a)

We have

$$\begin{aligned} f_x(x, y) =& 2x - 4 = 0 \quad \text{if} \quad x = 2; \\ f_y(x, y) =& 4y + 4 = 0 \quad \text{if} \quad y = -1. \end{aligned}$$

Critical point is $(2, -1)$. Since $f(x, y) \to \infty$ as $x^2 + y^2 \to \infty$. f has a local (and absolute) minimum value at that critical point.

b)

For critical points, we have

$$f_x = \sin(y) = 0, \quad f_y = x \cos(y) = 0.$$

Since $\sin(y)$ and $\cos(y)$ can not vanish at the same point, the only critical points correspond to $x = 0$ and $\sin(y) = 0$. They are $(0, n\pi)$ for all integers n. All are saddle points.

4 (13.1) *Old exam problem.*

Let $f(x, y) = (x^2 + y^2)e^x$.

a) Find and classify all critical points.

b) Find the tangent plane of the graph $z = f(x, y)$ at the point $(0, 1, 1)$.

Solution.

a)

(x, y) is a critical point $\iff \nabla f(x, y) = (0, 0)$. Then

$$\nabla f(x, y) = \Big(e^x(x^2 + y^2) + 2xe^x, 2ye^x\Big) = (0, 0).$$

This shows

$$\begin{aligned} & x^2 + y^2 + 2x = 0 \quad \text{and} \quad 2y = 0 \\ \iff \; & y = 0 \quad \text{and} \quad (x + 2)x = 0. \end{aligned}$$

We have two critical points: $(0,0)$ and $(-2,0)$. The Hessian matrix is

$$\mathbf{H}f(x,y) = \begin{pmatrix} e^x(x^2+y^2+2x+2x+2) & 2ye^x \\ 2ye^x & 2e^x \end{pmatrix}$$

Then

$$\begin{aligned}
\mathbf{H}f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad &\Longrightarrow \quad \det\Big(\mathbf{H}f(0,0)\Big) = 4 > 0, \\
\frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0 \quad &\Longrightarrow \quad (0,0) \quad \text{is a local minimum point;} \\
\mathbf{H}f(-2,0) = \begin{pmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{pmatrix} \quad &\Longrightarrow \quad \det\Big(\mathbf{H}f(-2,0)\Big) = -4e^{-4} < 0, \\
&\Longrightarrow \quad (-2,0) \quad \text{is a saddle point.}
\end{aligned}$$

b)

Since this is the case of explicit function, the tangent plane is given by $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, where $(a,b,c)=(-f_x,-f_y,1)$ and $(x_0,y_0,z_0)=(0,1,1)$. Thus,

$$\begin{aligned}
f_x = e^x(x^2+y^2)+2xe^x, &\quad f_x(0,1,1) = 1, \\
f_y = 2ye^x, &\quad f_y(0,1,1) = 2.
\end{aligned}$$

Then the tangent plane is given by

$$-x-2(y-1)+(z-1)=0 \quad \Longleftrightarrow \quad z = x+2y-1.$$

5 (13.1, 13.2)

a) Find the maximum and minimum values of $f(x,y)=xy-y^2$ on the disk $x^2+y^2\leq 1$.

b) Find the maximum and minimum values of $f(x,y)=\sin(x)\cos(y)$ on the closed triangular region bounded by the coordinate axes and the line $x+y=2\pi$.

How do you know that such extreme values must exist in **a)** and **b)**?

Solution.

a)

For critical points:

$$0 = f_x(x,y) = y, \quad 0 = f_y(x,y) = x-2y.$$

The only critical point is $(0,0)$, which lies inside $x^2+y^2\leq 1$. We have $f(0,0)=0$.

The boundary of $x^2 + y^2 \leq 1$ is the circle $x = \cos(t)$, $y = \sin(t)$, $-\pi \leq t \leq \pi$. On this circle, we have

$$\begin{aligned} g(t) =& f\big(\cos(t), \sin(t)\big) = \cos(t)\sin(t) - \sin^2(t) \\ =& \frac{1}{2}\Big[\sin(2t) + \cos(2t) - 1\Big], \quad (-\pi \leq t \leq \pi). \\ g(0) =& g(2\pi) = 0, \\ g'(t) =& \cos(2t) - \sin(2t). \end{aligned}$$

The critical points of g satisfy $\cos(2t) = \sin(2t)$, that is, $\tan(2t) = 1$, so $2t = \pm\frac{\pi}{4}$ or $\pm\frac{5\pi}{4}$, and $t = \pm\frac{\pi}{8}$ or $\pm\frac{5\pi}{8}$. We have

$$\begin{aligned} g\Big(\frac{\pi}{8}\Big) =& \frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{2} > 0, \\ g\Big(-\frac{\pi}{8}\Big) =& -\frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} = -\frac{1}{2}, \\ g\Big(\frac{5\pi}{8}\Big) =& -\frac{1}{2\sqrt{2}} - \frac{1}{2} - \frac{1}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} - \frac{1}{2}, \\ g\Big(-\frac{5\pi}{8}\Big) =& \frac{1}{2\sqrt{2}} - \frac{1}{2} - \frac{1}{2\sqrt{2}} = -\frac{1}{2}. \end{aligned}$$

Thus the maximum and minimum values of f on the disk $x^2 + y^2 \leq 1$ are $\frac{1}{\sqrt{2}} - \frac{1}{2}$ and $-\frac{1}{\sqrt{2}} - \frac{1}{2}$ respectively.

b)

Since $-1 \leq f(x, y) = \sin(x)\cos(y) \leq 1$ everywhere, and since $f(\frac{\pi}{2}, 0) = 1$, $f(\frac{3\pi}{2}, 0) = -1$, and both$(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$ belong to the triangle bounded by $x = 0$, $y = 0$ and $x + y = 2\pi$, therefore the maximum and minimum values of f over that triangle must be 1 and -1 respectively.

From the function f in **a)** and **b)**, we know that f is continuous on variables x and y whose domain are *closed* and *bounded* sets in $\mathbb{R}^2$, then the sufficient conditions for extreme values of Theorem 2 in Section 13.1 assures that the maximum and minimum values must exist.

6 (12.3, 12.6) *Old exam problem.*

Let

$$f(x, y) := \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that

a) f is continuous at $(0, 0)$,

b) $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

Solution.

a)

To show f is continuous at $(0,0)$, we need to show $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Indeed,

$$|f(x,y)| = \frac{|x|y^2}{x^2+y^2}, \tag{2}$$

and note that

$$|x|y^2 = |x||y||y| \le \frac{x^2+y^2}{2} \cdot |y|.$$

Taking this result into (2), we have

$$|f(x,y)| = \frac{|x|y^2}{x^2+y^2} \le \frac{\frac{x^2+y^2}{2}\cdot|y|}{x^2+y^2} = \frac{|y|}{2} \to 0, \quad \text{as} \quad (x,y)\to(0,0).$$

This means $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, which shows that f is continuous at $(0,0)$.

b)

By the definition of partial derivatives, we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{0}{h^2}-0}{h} = 0,$$

and similarly, we have

$$\frac{\partial f}{\partial y}(0,0) = 0.$$

If we want to show f is differentiable at $(0,0)$, by definition, we need to show

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = 0.$$

However, by $f(0,0)=0$, $f_x(0,0)=0$ and $f_y(0,0)=0$, we get

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{\frac{hk^2}{h^2+k^2}}{\sqrt{h^2+k^2}} = \lim_{\substack{k\to 0\\ h=k}} \frac{k^3}{2^{\frac{3}{2}}|k|^3} = \pm 2^{-\frac{3}{2}} \ne 0.$$

This means f is not differentiable at $(0,0)$.

7 (inverse function theorem)

Consider the following system of equations

$$u = x\cos(y) \quad \text{and} \quad v = 2x\sin(y).$$

Show that near x_0, y_0 with $x_0 \ne 0$, (x,y) can be expressed as differentiable function of (u,v) and compute $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ near (x_0,y_0).

Hint: check the conditions before using the inverse function theorem.

Solution.

Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ to be $\mathbf{F}(x, y) = (u, v) = \Big(x\cos(y), 2x\sin(y)\Big)$. To apply inverse function theorem to $\mathbf{F}$, we need to check two things.

1. $\mathbf{F}(x, y)$ is C^1 near (x_0, y_0).

Note that

$$\frac{\partial u}{\partial x} = \cos(y), \quad \frac{\partial u}{\partial y} = -x\sin(y), \quad \frac{\partial v}{\partial x} = 2\sin(y), \quad \frac{\partial v}{\partial y} = 2x\cos(y).$$

Clearly, all these partial derivatives exist and continuous everywhere. So the function is C^1 everywhere (in particular near (x_0, y_0)).

2. $\det D\mathbf{F}(x_0, y_0) \neq 0$.

$$\det D\mathbf{F}(x_0, y_0) = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\Bigg|_{(x_0,y_0)} = \begin{vmatrix} \cos(y_0) & -x_0\sin(y_0) \\ 2\sin(y_0) & 2x_0\cos(y_0) \end{vmatrix} = x_0 \neq 0.$$

Hence by the inverse function theorem, an C^1 inverse $\mathbf{F}^{-1}(u, v) = (x, y)$ exists and (x, y) can be expressed as differentiable function of (u, v).

So next, let us compute $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ at (u_0, v_0).

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}^{-1} = \begin{pmatrix} \cos(y_0) & -x_0\sin(y_0) \\ 2\sin(y_0) & 2x_0\cos(y_0) \end{pmatrix}^{-1}$$
$$= \frac{1}{2x_0}\begin{pmatrix} 2x_0\cos(y_0) & x_0\sin(y_0) \\ -2\sin(y_0) & \cos(y_0) \end{pmatrix}.$$

Hence, $\frac{\partial x}{\partial u}(u_0, v_0) = \cos(y_0)$ and $\frac{\partial x}{\partial v}(u_0, v_0) = \frac{1}{2}\sin(y_0)$.

8 (implicit function theorem)

Show that the equations

$$\begin{cases} xy^2 + zu + v^2 = 3, \\ x^3z + 2y - uv = 2, \\ xu + yv - xyz = 1, \end{cases} \tag{3}$$

can be solved for x, y, and z as functions of u and v near the point P_0 where $(x, y, z, u, v) = (1, 1, 1, 1, 1)$, and find $\frac{\partial y}{\partial u}$ at $(u, v) = (1, 1)$.

Hint: check the conditions before using the implicit function theorem.

Solution.

Define $\mathbf{F} : \mathbb{R}^5 \to \mathbb{R}^3$ to be

$$\mathbf{F}(x,y,z,u,v) = (xy^2 + zu + v^2 - 3, x^3z + 2y - uv - 2, xu + yv - xyz - 1) = (f_1, f_2, f_3).$$

Step 1: Check $\mathbf{F}$ is $\mathbf{C^1}$ near $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$.

A quick differentiation yields

$$\begin{aligned}
&\frac{\partial f_1}{\partial x} = y^2, \quad \frac{\partial f_1}{\partial y} = 2xy, \quad \frac{\partial f_1}{\partial z} = u, \quad \frac{\partial f_1}{\partial u} = z, \quad \frac{\partial f_1}{\partial v} = 2v;\\
&\frac{\partial f_2}{\partial x} = 3x^2z, \quad \frac{\partial f_2}{\partial y} = 2, \quad \frac{\partial f_2}{\partial z} = x^3, \quad \frac{\partial f_2}{\partial u} = -v, \quad \frac{\partial f_2}{\partial v} = -u;\\
&\frac{\partial f_3}{\partial x} = u - yz, \quad \frac{\partial f_3}{\partial y} = v - xz, \quad \frac{\partial f_3}{\partial z} = -xy, \quad \frac{\partial f_3}{\partial u} = x, \quad \frac{\partial f_3}{\partial v} = y.
\end{aligned}$$

Since all partial derivatives exist and continuous everywhere, $\mathbf{F}$ is C^1 everywhere (in particular near $(1,1,1,1,1)$).

Step 2: Check $\mathbf{F}(P_0) = \mathbf{0}$.

Clearly $\mathbf{F}(1,1,1,1,1) = (0,0,0)$.

Step 3: Check $\det \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}(P_0) \neq 0$.

$$\begin{aligned}
\det \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}(P_0) &= \det \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}\Bigg|_{(1,1,1,1,1)} \\
&= \det \begin{pmatrix} y^2 & 2xy & u \\ 3x^2z & 2 & x^3 \\ u - yz & v - xz & -xy \end{pmatrix}\Bigg|_{(1,1,1,1,1)} \\
&= \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 4 \neq 0.
\end{aligned}$$

Hence, by the implicit function theorem, (x,y,z) can be expressed as a differentiable function of (u,v) near $(1,1,1,1,1)$.

To find $\frac{\partial y}{\partial u}$ (denote by y_u), we differentiate both sides of each equation in (3) with respect to u (i.e. implicit differentiation) to get

$$\begin{cases} xy^2 + zu + v^2 = 3, \\ x^3z + 2y - uv = 2, \\ xu + yv - xyz = 1, \end{cases} \implies \begin{cases} y^2x_u + 2xyy_u + uz_u + z = 0, \\ 3x^2zx_u + 2y_u + x^3z_u - v = 0, \\ (u - yz)x_u + (v - xz)y_u - xyz_u + x = 0. \end{cases}$$

Taking $(1,1,1,1,1)$ into the above system, we get

$$\begin{cases} x_u + 2y_u + z_u = -1, \\ 3x_u + 2y_u + z_u = 1, \\ - z_u = -1, \end{cases}$$

which gives,

$$x_u(P_0) = 1, \quad y_u(P_0) = -\frac{3}{2}, \quad z_u(P_0) = 1 \quad \implies \quad \frac{\partial y}{\partial u}(P_0) = -\frac{3}{2}.$$