Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1101 Basic Calculus I
Spring 2022

Exercise set 5: Solutions

1 (12.4)

Find all the second partial derivatives of the given functions below.

a) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2(1 + y^2)$

b) $g\colon \left\{(x, y) \in \mathbb{R}^2 \,|\, xy \neq -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}\right\} \to \mathbb{R}$, $(x, y) \mapsto \ln\left(1 + \sin(xy)\right)$

Solution.

a)

Direct computations shows that

$$\frac{\partial f}{\partial x} = 2x(1+y^2), \quad \frac{\partial f}{\partial y} = 2x^2 y, \quad \frac{\partial^2 f}{\partial x^2} = 2(1+y^2), \quad \frac{\partial^2 f}{\partial y^2} = 2x^2,$$
$$\frac{\partial^2 f}{\partial x \partial y} = 4xy = \frac{\partial^2 f}{\partial y \partial x}.$$

b)

Direct computations shows that

$$\frac{\partial g}{\partial x} = \frac{y\cos(xy)}{1+\sin(xy)}, \quad \frac{\partial g}{\partial y} = \frac{x\cos(xy)}{1+\sin(xy)},$$
$$\frac{\partial^2 g}{\partial x^2} = \frac{(1+\sin(xy))\big(-y^2\sin(xy)\big) - \big(y\cos(xy)\big)\big(y\cos(xy)\big)}{\big(1+\sin(xy)\big)^2} = -\frac{y^2}{1+\sin(xy)},$$
$$\frac{\partial^2 g}{\partial y^2} = -\frac{x^2}{1+\sin(xy)} \quad \text{(by symmetry)},$$
$$\frac{\partial^2 g}{\partial x \partial y} = \frac{(1+\sin(xy))\big(\cos(xy) - xy\sin(xy)\big) - \big(y\cos(xy)\big)\big(x\cos(xy)\big)}{\big(1+\sin(xy)\big)^2}$$
$$= \frac{\cos(xy) - xy}{1+\sin(xy)} = \frac{\partial^2 g}{\partial y \partial x}.$$

2 (12.7)

Use definition to find the directional derivative of $f(x, y) = x^2 + xy$ in the direction $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ at point $(1, 2)$.

Solution.

By definition, along the direction $\mathbf{u} = (u, v) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, at point $(a, b) = (1, 2)$, we have

$$\begin{aligned}
D_{\mathbf{u}} f(1,2) &= \lim_{h \to 0+} \frac{f(a+hu, b+hv) - f(a,b)}{h} \\
&= \lim_{h \to 0+} \frac{f\left(1 + \frac{\sqrt{2}}{2}h, 2 + \frac{\sqrt{2}}{2}h\right) - f(1,2)}{h} \\
&= \lim_{h \to 0+} \frac{\left(1 + \frac{\sqrt{2}}{2}h\right)^2 + \left(1 + \frac{\sqrt{2}}{2}h\right)\left(2 + \frac{\sqrt{2}}{2}h\right) - 3}{h} \\
&= \lim_{h \to 0+} \frac{1 + \sqrt{2}h + \frac{h^2}{2} + 2 + \frac{3\sqrt{2}}{2}h + \frac{h^2}{2} - 3}{h} \\
&= \lim_{h \to 0+} \frac{\frac{5\sqrt{2}}{2}h + h^2}{h} \\
&= \frac{5\sqrt{2}}{2}.
\end{aligned}$$

3 (12.9)

Find the Taylor series for the given functions near the indicated points.

a) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 + xy + y^3$, $\quad (1, -1)$

b) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto e^{x^2+y^2}$, $\quad (0, 0)$

Solution.

a)

Let $u = x - 1$, $v = y + 1$. Thus,

$$\begin{aligned}
f(x,y) =& x^2 + xy + y^3 \\
=& (u+1)^2 + (u+1)(v-1) + (v-1)^3 \\
=& 1 + 2u + u^2 - 1 + v - u + uv + v^3 - 3v^2 + 3v - 1 \\
=& -1 + u + 4v + u^2 + uv - 3v^2 + v^3 \\
=& -1 + (x-1) + 4(y+1) + (x-1)^2 + (x-1)(y+1) - 3(y+1)^2 + (y+1)^3.
\end{aligned}$$

This is the Taylor series for f about $(1, -1)$.

b)

Recall that the Taylor series for e^x about 0 is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then we have

$$\begin{aligned} f(x,y) =& e^{x^2+y^2} \\ =& \sum_{n=0}^{\infty} \frac{(x^2+y^2)^n}{n!} \\ =& \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{2j} y^{2n-2j} \\ =& \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^{2j} y^{2n-2j}}{j!(n-j)!}, \end{aligned} \tag{1}$$

where we have used the binomial theorem in the second last step that

$$\begin{aligned} &(x^2+y^2)^n \\ =& \binom{n}{0} x^{2n} y^0 + \binom{n}{1} x^{2n-2} y^2 + \binom{n}{2} x^{2n-4} y^4 + \cdots + \binom{n}{n-1} x^2 y^{2n-2} + \binom{n}{n} x^0 y^{2n} \\ =& \sum_{j=0}^{n} \binom{n}{j} x^{2n-2j} y^{2j} = \sum_{j=0}^{n} \binom{n}{j} x^{2j} y^{2n-2j}. \end{aligned}$$

Thus, (1) is the Taylor series for f about $(0,0)$.

4 (12.8)

If $x = u^3 + v^3$ and $y = uv - v^2$ are solved for u and v in terms of x and y, evaluate

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial (u,v)}{\partial (x,y)}$$

at the point $(u,v) = (1,1)$.

Solution.

From the expression of x and y, we know that u, v are the functions of variables x, y. Taking partial derivative with respect to x to the given equations, we have

$$\begin{aligned} 1 =& 3u^2 \frac{\partial u}{\partial x} + 3v^2 \frac{\partial v}{\partial x}, \\ 0 =& v \frac{\partial u}{\partial x} + (u-2v) \frac{\partial v}{\partial x}. \end{aligned}$$

At $u = v = 1$, we have

$$\begin{aligned} 1 =& 3 \frac{\partial u}{\partial x} + 3 \frac{\partial v}{\partial x}, \\ 0 =& \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}. \end{aligned}$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{1}{6}$.

Similarly, taking partial derivative with respect to y to the given equations and putting $u = v = 1$, we have

$$\begin{aligned} 0 =& 3\frac{\partial u}{\partial y} + 3\frac{\partial v}{\partial y}, \\ 1 =& \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}. \end{aligned}$$

Thus, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y} = \frac{1}{2}$.

Finally,

$$\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{(u,v)=(1,1)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = -\frac{1}{6}.$$

5 (12.7)

Let $f(x, y, z) = x^3 - xy^2 - z$, $P_0 = (1, 1, 0)$.

a) In what direction at P_0 does f increase most rapidly? What is the rate of increase of f in that direction?

b) In what direction at P_0 does f decrease most rapidly? What is the rate of decrease of f in that direction?

Solution.

The gradient of f is given by

$$\nabla f(x, y, z) = (3x^2 - y^2, -2xy, -1), \quad \nabla f(1, 1, 0) = (2, -2, -1).$$

a)

f increases most rapidly at P_0 along the direction $\nabla f(1, 1, 0)$, and the rate of increase is

$$|\nabla f(1, 1, 0)| = \sqrt{2^2 + (-2)^2 + 1} = 3.$$

b)

f decreases most rapidly at P_0 along the direction $-\nabla f(1, 1, 0)$, and the rate of decrease is

$$-|\nabla f(1, 1, 0)| = -\sqrt{2^2 + (-2)^2 + 1} = -3.$$

6 (12.7)

Find an equation of the curve in the xy-plane that passes through the point $(1, 1)$ and intersects all level curves of the function $f(x, y) = x^4 + y^2$ at right angles.

Solution.

Let the curve be $y = g(x)$. At (x, y) this curve has normal $\nabla\Big(g(x) - y\Big) = \Big(g'(x), -1\Big)$.

A curve of the family $x^4 + y^2 = C$ has normal $\nabla(x^4 + y^2) = (4x^3, 2y)$.

These curves will intersect at right angles if their normals are perpendicular. Thus we require that

$$0 = 4x^3 g'(x) - 2y = 4x^3 g'(x) - 2g(x),$$

or, equivalently,

$$\frac{g'(x)}{g(x)} = \frac{1}{2x^3}.$$

Integration gives $\ln(|g(x)|) = -\dfrac{1}{4x^2} + \ln(|C|)$, or $g(x) = Ce^{-\frac{1}{4x^2}}$.

Since the curve passes through $(1, 1)$, we must have $1 = g(1) = Ce^{-\frac{1}{4}}$, so $C = e^{\frac{1}{4}}$.

The required curve is $y = e^{\frac{1}{4} - \frac{1}{4x^2}}$.

7 (12.8)

Evaluate the Jacobian $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ for the transformation to polar coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$. Near what points (r, θ) is the transformation one-to-one (hence invertible)?

Solution.

From the expression of $x = r\cos(\theta)$, $y = r\sin(\theta)$, we have

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r.$$

The transformation is one-to-one (and hence invertible) near any point where $r \neq 0$, that is, near any point except the origin.

8 (12.4)

Let

$$F(x, y) = \begin{cases} \dfrac{2xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

a) Calculate $F_x(x, y)$, $F_y(x, y)$, $F_{xy}(x, y)$, and $F_{yx}(x, y)$ at points $(x, y) \neq (0, 0)$. Also calculate these derivatives at $(0, 0)$.

b) Observe that $F_{yx}(0,0) = 2$ and $F_{xy}(0,0) = -2$. Does this result contradict Schwarz's theorem? Explain why.

Hint: Note the continuity assumption in Schwarz's theorem on the second partial derivatives of the function.

Solution.

a)

Let

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

For $(x,y) \neq (0,0)$, we have

$$\begin{aligned} f_x(x,y) =& \frac{(x^2+y^2)2y - 2xy(2x)}{(x^2+y^2)^2} = \frac{2y(y^2-x^2)}{(x^2+y^2)^2}, \\ f_y(x,y) =& \frac{2x(x^2-y^2)}{(x^2+y^2)^2} \quad \text{(by symmetry)}. \end{aligned}$$

Let $F(x,y) = (x^2-y^2)f(x,y)$. Then we calculate

$$\begin{aligned} F_x(x,y) =& 2xf(x,y) + (x^2-y^2)f_x(x,y) = 2xf(x,y) - \frac{2y(y^2-x^2)^2}{(x^2+y^2)^2}, \\ F_y(x,y) =& -2yf(x,y) + (x^2-y^2)f_y(x,y) = -2yf(x,y) + \frac{2x(x^2-y^2)^2}{(x^2+y^2)^2}, \\ F_{xy}(x,y) =& \frac{2(x^6+9x^4y^2-9x^2y^4-y^6)}{(x^2+y^2)^3} = F_{yx}(x,y). \end{aligned} \tag{2}$$

For the values at $(0,0)$, we revert to the definition of derivative to calculate the partials that

$$\begin{aligned} F_x(0,0) &= \lim_{h\to 0} \frac{F(h,0)-F(0,0)}{h} = 0 = F_y(0,0), \\ F_{xy}(0,0) &= \lim_{k\to 0} \frac{F_x(0,k)-F_x(0,0)}{k} = \lim_{k\to 0} \frac{-2k(k^4)}{k(k^4)} = -2, \\ F_{yx}(0,0) &= \lim_{h\to 0} \frac{F_y(h,0)-F_y(0,0)}{h} = \lim_{h\to 0} \frac{2h(h^4)}{h(h^4)} = 2. \end{aligned}$$

b) This does not contradict Schwarz's theorem since the partials F_{yx} and F_{xy} are not continuous at $(0,0)$. For instance, from (2), we have

$$\begin{aligned} \lim_{\substack{x\to 0 \\ y=x}} F_{xy}(x,y) &= \lim_{x\to 0} \frac{2(x^6+9x^6-9x^6-x^6)}{(x^2+x^2)^3} = 0, \\ \lim_{\substack{x\to 0 \\ y=0}} F_{xy}(x,y) &= \lim_{x\to 0} \frac{2(x^6)}{x^6} = 2. \end{aligned}$$

When (x, y) tends to $(0, 0)$ in the above two different ways, the limits are not equal. So F_{xy} is not continuous at $(0, 0)$. We can show F_{yx} is not continuous at $(0, 0)$ in the similar way.

9 (12.3, 12.6)

Let

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is continuous at point $(0, 0)$ and the first order partial derivatives $f_x(0, 0)$, $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

Solution.

Since

$$0 \leq \frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}} \leq \frac{(x^2+y^2)^2}{(x^2+y^2)^{\frac{3}{2}}} = \sqrt{x^2+y^2},$$

and

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0,$$

we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Since $f(0, 0) = 0$, we get $\lim\limits_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$, that is, $f(x, y)$ is continuous at point $(0, 0)$.

A direct computation shows that

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0, \tag{3}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0. \tag{4}$$

If we want to show f is differentiable at $(0, 0)$, by definition, we need to show

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} = 0.$$

However, by $f(0, 0) = 0$, (3) and (4), we get

$$\begin{aligned} &\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{\sqrt{h^2+k^2}} \\ =& \lim_{(h,k)\to(0,0)} \frac{\frac{h^2k^2}{(h^2+k^2)^{\frac{3}{2}}}}{\sqrt{h^2+k^2}} = \lim_{\substack{h\to 0 \\ k=h}} \frac{h^4}{4h^4} = \frac{1}{4} \neq 0. \end{aligned}$$

This means f is not differentiable at $(0, 0)$.

Finally, we have shown that f is continuous at point $(0, 0)$ and the first order partial derivatives $f_x(0, 0)$, $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.