MA1103 Vector Calculus
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Exercise set 3: Solutions

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1 Compute $\det(A+B)$ and $\det(AB)$ for

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

$$\det(A+B) = \begin{vmatrix} 0 & 2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 0, \quad \det(AB) = \begin{vmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -2.$$

2 For what values of the parameter λ is the length $s(T)$ of the curve $\mathbf{r} = (t, \lambda t^2, t^3)$ $(0 \leq t \leq T)$ given by $s(T) = T + T^3$?

Solution.

We have

$$v = \sqrt{1 + (2\lambda t)^2 + 9t^4} = \sqrt{(1+3t^2)^2} \quad \text{(speed)}$$

if $4\lambda^2 = 6$, that is, if $\lambda = \pm\sqrt{\frac{3}{2}}$. In this case, the length of the curve is

$$s(T) = \int_0^T (1+3t^2)\, dt = T + T^3.$$

3 Find the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ for the curve (1) at the point indicated.

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k} \qquad \text{at} \quad (1,1,1), \tag{1}$$

where $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0), \mathbf{k} = (0,0,1)$.

Solution.

$$\begin{aligned}\mathbf{r} =& t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}\\ \mathbf{v} =& \mathbf{i} + 2t\mathbf{j} + \mathbf{k}\\ \mathbf{a} =& 2\mathbf{j}\\ \mathbf{v}\times\mathbf{a} =& -2\mathbf{i} + 2\mathbf{k}.\end{aligned}$$

At $(1,1,1)$, where $t = 1$, we have

$$\begin{aligned}\mathbf{T} =& \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{6}}\\ \mathbf{B} =& \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{-(\mathbf{i}-\mathbf{k})}{\sqrt{2}}\\ \mathbf{N} =& \mathbf{B}\times\mathbf{T} = \frac{-(\mathbf{i}-\mathbf{j}+\mathbf{k})}{\sqrt{3}}.\end{aligned}$$

4 Let $\mathbf{u}, \mathbf{v}$ be two given vectors in a (real) vector space V equipped with an inner product $\langle,\rangle_V$, and let $S(\mathbf{v})$ denote the (real) linear span of $\mathbf{v}$. The **projection of u on v**, $P_{\mathbf{v}}(\mathbf{u})$, is the point in $S(\mathbf{v})$ closest to $\mathbf{u}$.

a) Determine a formula for $P_{\mathbf{v}}(\mathbf{u})$.

b) Find $P_{(1,0,-2)}(1,2,3)$.

Solution.

a) We have $\langle \mathbf{u} - P_{\mathbf{v}}(\mathbf{u}), \mathbf{v}\rangle_V = 0$. Since $P_{\mathbf{v}}(\mathbf{u}) \in S$ we can write $P_{\mathbf{v}}(\mathbf{u}) = a\mathbf{v}$ for some scalar a and it remains to determine a. The previous condition becomes

$$0 = \langle \mathbf{u} - a\mathbf{v}, \mathbf{v}\rangle_V = \langle \mathbf{u}, \mathbf{v}\rangle_V - a\langle \mathbf{v}, \mathbf{v}\rangle_V,$$

which gives $a = \frac{\langle \mathbf{u},\mathbf{v}\rangle_V}{\langle \mathbf{v},\mathbf{v}\rangle_V}$. Thus we get the formula

$$P_{\mathbf{v}}(\mathbf{u}) = \frac{\langle \mathbf{u},\mathbf{v}\rangle_V}{\langle \mathbf{v},\mathbf{v}\rangle_V}\mathbf{v}.$$

b) Using a) with $V = \mathbb{R}^3$ and $\langle,\rangle_V$ the usual inner product on $\mathbb{R}^3$, we compute and find

$$\begin{aligned}(1,2,3)\cdot(1,0,-2) &= 1-6 = -5\\ (1,0,-2)\cdot(1,0,-2) &= 1+4 = 5\\ P_{(1,0,-2)}(1,2,3) &= -(1,0,-2) = (-1,0,2).\end{aligned}$$

5 Specify the natural domain of the function below.

$$f\colon (x,y) \mapsto \arcsin(x+y)$$

Solution.

The domain consists of all points in the strip $-1 \leq x + y \leq 1$.

6 Determine in each case below whether the limit exists, and in the case it does, give its value.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$

b) $\displaystyle\lim_{(x,y)\to(1,\pi)} \frac{\cos(xy)}{1-x-\cos(y)}$

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$

Solution.

a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$ does not exist.

If $(x,y) \to (0,0)$ along $x = 0$, then $\frac{x^2+y^2}{y} = y \to 0$.

If $(x,y) \to (0,0)$ along $y = x^2$, then $\frac{x^2+y^2}{y} = 1 + x^2 \to 1$.

b) $\displaystyle\lim_{(x,y)\to(1,\pi)} \frac{\cos(xy)}{1-x-\cos(y)} = \frac{\cos(\pi)}{1-1-\cos(\pi)} = -1.$

c) If $x = 0$ and $y \neq 0$, then $\frac{x^2y^2}{2x^4+y^4} = 0$.

If $x = y \neq 0$, then $\frac{x^2y^2}{2x^4+y^4} = \frac{x^4}{2x^4+x^4} = \frac{1}{3}$.

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$ does not exist.

7 How can the function

$$f(x,y) = \frac{x^3-y^3}{x-y}, \qquad (x \neq y),$$

be defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane?

Solution.

For $x \neq y$, we have

$$f(x,y) = \frac{x^3-y^3}{x-y} = x^2 + xy + y^2.$$

The latter expression has the value $3x^2$ at points of the line $x = y$. Therefore, if we extend the definition of $f(x, y)$ so that $f(x, x) = 3x^2$, then the resulting function will be equal to $x^2 + xy + y2$ everywhere, and so continuous everywhere.

8 Find all first order partial derivatives of the given functions below.

a) $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$
b) $f(x, y, z) = x^3y^4z^5$
c) $f(x, y, z) = \ln(1 + e^{xyz})$

Solution.

a) $\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2 + y^2)^{-\frac{3}{2}}(2x) = -\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$.
By symmetry, $\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}$.

b) $\frac{\partial f}{\partial x} = 3x^2y^4z^5$,
$\frac{\partial f}{\partial y} = 4x^3y^3z^5$,
$\frac{\partial f}{\partial z} = 5x^3y^4z^4$.

c) $\frac{\partial f}{\partial x} = \frac{yze^{xyz}}{1+e^{xyz}}$,
$\frac{\partial f}{\partial y} = \frac{xze^{xyz}}{1+e^{xyz}}$,
$\frac{\partial f}{\partial z} = \frac{xye^{xyz}}{1+e^{xyz}}$.

9 Define a function

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

(1) Calculate $f_x(x, y)$ and $f_y(x, y)$ at all points (x, y) in the plane.
(2) Is f continuous at $(0, 0)$? Are f_x and f_y continuous at $(0, 0)$?

Solution.

If $(x, y) \neq (0, 0)$, then

$$\begin{aligned} f_x(x,y) =& \frac{(x^2+y^2)3x^2-(x^3-y^3)2x}{(x^2+y^2)^2} \\ =& \frac{x^4+3x^2y^2}{(x^2+y^2)^2}, \\ f_y(x,y) =& \frac{(x^2+y^2)(-3y^2)-(x^3-y^3)2x}{2}y \\ =& -\frac{y^4+3x^2y^2+2x^3y}{(x^2+y^2)^2} \end{aligned}$$

Also, at $(0, 0)$,

$$f_x(0,0) = \lim_{h\to 0} \frac{h^3}{h \cdot h^2} = 1, \quad f_y(0,0) = \lim_{k\to 0} \frac{-k^3}{k \cdot k^2} = -1.$$

Neither f_x nor f_y has a limit at $(0, 0)$ (the limits along $x = 0$ and $y = 0$ are different in each case), so neither function is continuous at $(0, 0)$. However, f is continuous at $(0, 0)$ because

$$|f(x,y)| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \leq |x| + |y|,$$

which $\to 0$ as $(x, y) \to (0, 0)$.