Exercise set 11: Solutions

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1 (16.1)

Calculate the divergence and curl of $\mathbf{F}$ for the given vector fields on variables x, y and z.

a) $\mathbf{F}(x, y, z) = (yz, xz, xy)$.

b) $\mathbf{F}(r, \theta) = \Big(r, \sin(\theta)\Big)$, where (r, θ) are polar coordinates in the plane.

Hint: Use $x = r\cos(\theta)$, $y = r\sin(\theta)$.

Solution.

a)

By the definition of divergence and curl, we get

$$\operatorname{div}\mathbf{F} = (yz)_x + (xz)_y + (xy)_z = 0,$$

and

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = (0, 0, 0).$$

b)

Since $x = r\cos(\theta)$, and $y = r\sin(\theta)$, we have

$$r^2 = x^2 + y^2.$$

Taking the partial derivative on x to the left and right side of the above equation, we get

$$2r \cdot r_x = 2x, \quad \Longrightarrow \quad r_x = \frac{x}{r} = \cos(\theta), \tag{1}$$

where we have used $x = r\cos(\theta)$ in the last step. Similarly, using $y = r\sin(\theta)$, we have

$$2r \cdot r_y = 2y, \quad \Longrightarrow \quad r_y = \frac{y}{r} = \sin(\theta). \tag{2}$$

Then by $\sin(\theta) = \dfrac{y}{r}$ and $r_y = \sin(\theta)$ in (2), we get

$$\frac{\partial}{\partial y}\sin(\theta) = \frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{r - y \cdot r_y}{r^2} = \frac{r - r\sin(\theta) \cdot \sin(\theta)}{r^2} = \frac{\cos^2(\theta)}{r}. \tag{3}$$

Thus, using (1) and (3), the divergence is given by

$$\operatorname{div}\mathbf{F} = r_x + \Big(\sin(\theta)\Big)_y = \cos(\theta) + \frac{\cos^2(\theta)}{r}.$$

Next, by the definition of curl, we have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ r & \sin(\theta) & 0 \end{vmatrix} = [\sin(\theta)]_x\mathbf{k} - r_y\mathbf{k} = \Big(0, 0, [\sin(\theta)]_x - r_y\Big). \tag{4}$$

We have computed r_y in (2), so let us compute $[\sin(\theta)]_x$ next. By $\sin(\theta) = \frac{y}{r}$ and $r_x = \cos(\theta)$ in (1), we get

$$\frac{\partial}{\partial x}\sin(\theta) = \frac{\partial}{\partial x}\Big(\frac{y}{r}\Big) = y \cdot (-1)r^{-2} \cdot r_x = -r\sin(\theta) \cdot \frac{1}{r^2} \cdot \cos(\theta) = -\frac{\sin(\theta)\cos(\theta)}{r}. \tag{5}$$

Finally, substituting (5) and (2) into (4), we get

$$\operatorname{curl}\mathbf{F} = \left(0, 0, -\frac{\sin(\theta)\cos(\theta)}{r} - \sin(\theta)\right).$$

2 (15.5)

Find

$$\iint_D y\,dS,$$

where D is the part of the plane $z = 1 + y$ that lies inside the cone $z = \sqrt{2(x^2+y^2)}$.

Solution.

The intersection of the plane $z = 1 + y$ and the cone $z = \sqrt{2(x^2+y^2)}$ has projection onto the xy-plane the elliptic disk E bounded by

$$\begin{aligned} (1+y)^2 &= 2(x^2+y^2), \\ 1 + 2y + y^2 &= 2x^2 + 2y^2, \\ 2x^2 + y^2 - 2y + 1 &= 2, \\ x^2 + \frac{(y-1)^2}{2} &= 1, \end{aligned}$$

with centroid $(0, 1)$. If D is the part of the plane lying inside the cone, then the area element on D is

$$dS = \sqrt{1 + \Big(\frac{\partial z}{\partial y}\Big)^2}\,dxdy = \sqrt{2}\,dxdy.$$

Thus,

$$\iint_D y\,dS = \sqrt{2}\iint_E y\,dxdy := \sqrt{2}I. \tag{6}$$

Let us concentrate on computing I next.

Fix any $x \in [-1,1]$ in E, then we find the range of y is from $1-\sqrt{2-2x^2}$ to $1+\sqrt{2-2x^2}$. Thus, we have

$$\begin{aligned}
I &= \iint_E y\,dxdy \\
&= \int_{-1}^{1}\left(\int_{1-\sqrt{2-2x^2}}^{1+\sqrt{2-2x^2}} y\,dy\right)dx \\
&= \int_{-1}^{1}\left(\frac{1}{2}y^2\Big|_{1-\sqrt{2-2x^2}}^{1+\sqrt{2-2x^2}}\right)dx \\
&= \frac{1}{2}\int_{-1}^{1}\left[\left(1+\sqrt{2-2x^2}\right)^2-\left(1-\sqrt{2-2x^2}\right)^2\right]dx \\
&= \frac{1}{2}\int_{-1}^{1} 2\cdot 2\sqrt{2-2x^2}\,dx \\
&= 2\int_{-1}^{1}\sqrt{2-2x^2}\,dx.
\end{aligned}$$

Let $x=\sin(t)$, then when $x=-1$, $t=-\frac{\pi}{2}$, when $x=1$, $t=\frac{\pi}{2}$, and $dx=\cos(t)\,dt$. Then

$$\begin{aligned}
I &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{2}\cdot\cos(t)\cdot\cos(t)\,dt \\
&= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\cos(2t)+1\Big)\,dt \\
&= \sqrt{2}\pi,
\end{aligned}$$

where the integration of $\cos(2t)$ vanishes. Thus, substituting I into (6), we finally get

$$\iint_D y\,dS = \sqrt{2}\cdot\sqrt{2}\pi = 2\pi.$$

3 (15.6)

Find the flux of $\mathbf{F}=(yz,-xz,x^2+y^2)$ upward through the surface

$$\mathbf{r}=\Big(e^u\cos(v),e^u\sin(v),u\Big),$$

where $0\le u\le 1$ and $0\le v\le \pi$.

Solution.

The aimed area D: $\mathbf{r}=\Big(e^u\cos(v),e^u\sin(v),u\Big)$, $(0\le u\le 1, 0\le v\le \pi)$, has upward surface element

$$\begin{aligned}
\mathbf{N}\,dS =& \frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\,dudv \\
=& \Big(-e^u\cos(v),-e^u\sin(v),e^{2u}\Big)\,dudv.
\end{aligned}$$

The flux of $\mathbf{F} = (yz, -xz, x^2 + y^2)$ upward through D is

$$\begin{aligned} &\iint_D \mathbf{F} \cdot \mathbf{N}\, dS \\ =& \int_0^1 \left[\int_0^\pi \Big(ue^u \sin(v), -ue^u \cos(v), e^{2u} \Big) \cdot \Big(-e^u \cos(v), -e^u \sin(v), e^{2u} \Big)\, dv \right] du \\ =& \int_0^1 \left[\int_0^\pi \Big(-ue^{2u} \sin(v)\cos(v) + ue^{2u} \sin(v)\cos(v) + e^{4u} \Big)\, dv \right] du. \end{aligned} \tag{7}$$

Let us compute the inner integration one by one. Firstly,

$$\begin{aligned} &\int_0^\pi -ue^{2u} \sin(v)\cos(v)\, dv \\ =& -ue^{2u} \int_0^\pi \sin(v)\, d\Big(\sin(v)\Big) \\ =& -ue^{2u} \cdot \frac{1}{2} \sin^2(v) \Big|_0^\pi \\ =& 0. \end{aligned}$$

Using the above computational result, the second integration is also given by

$$\int_0^\pi ue^{2u} \sin(v)\cos(v)\, dv = 0.$$

Thirdly,

$$\int_0^\pi e^{4u}\, dv = e^{4u}\pi.$$

Thus, substituting the above three results into (7), we get

$$\iint_D \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1 e^{4u}\pi\, du = \frac{\pi}{4} e^{4u} \Big|_0^1 = \frac{\pi}{4}(e^4 - 1).$$

4 (15.6)

Let S be the part of the surface of the cylinder $y^2 + z^2 = 16$ that lies in the first octant and between the planes $x = 0$ and $x = 5$. Find the flux of $(3z^2x, -x, -y)$ away from the x-axis through S.

Solution.

Rewrite $y^2 + z^2 = 16$ as $z = \sqrt{16 - y^2} := f(x, y)$. The first octant part of the cylinder $y^2 + z^2 = 16$ has outward vector surface element

$$\begin{aligned} \mathbf{N}\, dA = (-f_x. - f_y, 1)\, dxdy &= \Big(0, -\frac{1}{2}(16 - y^2)^{-\frac{1}{2}} \cdot (-2y), 1\Big)\, dxdy \\ &= \Big(0, \frac{y}{\sqrt{16 - y^2}}, 1\Big)\, dxdy. \end{aligned}$$

The flux of $(3z^2x, -x, -y)$ outward through the specified surface S is

$$\begin{aligned}
&\iint_S \mathbf{F}\cdot\mathbf{N}\,dA\\
=&\int_0^5\left[\int_0^4 (3z^2x,-x,-y)\cdot\Big(0,\frac{y}{\sqrt{16-y^2}},1\Big)\,dy\right]dx\\
=&\int_0^5\left[\int_0^4 \Big(0-\frac{xy}{\sqrt{16-y^2}}-y\Big)\,dy\right]dx. \qquad (8)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^4 -\frac{xy}{\sqrt{16-y^2}}\,dy\\
=&\frac{x}{2}\int_0^4 (16-y^2)^{-\frac{1}{2}}\,d\Big(16-y^2\Big)\\
=&\frac{x}{2}\cdot\frac{(16-y^2)^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_0^4\\
=&-4x.
\end{aligned}$$

taking this result into (8), we have

$$\begin{aligned}
&\iint_S \mathbf{F}\cdot\mathbf{N}\,dA\\
=&\int_0^5\Big(-4x-\frac{y^2}{2}\Bigg|_0^4\Big)\,dx\\
=&\int_0^5(-4x-8)\,dx\\
=&(-2x^2-8x)\Bigg|_0^5\\
=&-90.
\end{aligned}$$

5 (15.4)

For what values of the constants a, b, and c can you determine the value of the integral I of the tangential component of

$$\mathbf{F}=\Big(axy+3yz, x^2+3xz+by^2z, bxy+cy^3\Big)$$

along a curve from $(0,1,-1)$ to $(2,1,1)$ without knowing the curve? What is the value of the integral?

Solution.

$\int_C \mathbf{F}\cdot d\mathbf{r}$ can be determined using only the endpoints of C, provided

$$\mathbf{F}=\Big(axy+3yz, x^2+3xz+by^2z, bxy+cy^3\Big)$$

is conservative, that is, if

$$\begin{aligned} ax + 3z = \frac{\partial F_1}{\partial y} &= \frac{\partial F_2}{\partial x} = 2x + 3z, \\ 3y = \frac{\partial F_1}{\partial z} &= \frac{\partial F_3}{\partial x} = by, \\ 3x + by^2 = \frac{\partial F_2}{\partial z} &= \frac{\partial F_3}{\partial y} = bx + 3cy^2. \end{aligned}$$

Thus we need $a = 2$, $b = 3$, and $c = 1$. With these values, let us find a potential of $\mathbf{F}$, where

$$\mathbf{F} = \Big(2xy + 3yz, x^2 + 3xz + 3y^2z, 3xy + y^3\Big).$$

Observing the first position of $\mathbf{F}$, we have

$$\int (2xy + 3yz)\, dx = x^2y + 3xyz.$$

We can suppose the potential function as

$$f(x, y, z) = x^2y + 3xyz + C_1, \tag{9}$$

where C_1 is a function that depends on the variables y and z. So we now know that

$$f_x = 2xy + 3yz.$$

Let us continue to consider the second position of $\mathbf{F}$. From our result of f in (9), we have

$$f_y = x^2 + 3xz + (C_1)_y.$$

Comparing this result with the second position of $\mathbf{F}$, we get

$$(C_1)_y = 3y^2z, \quad \Longrightarrow \quad C_1 = y^3z + C_2,$$

where C_2 is a function that depends on the variable z. So by now we have

$$f(x, y, z) = x^2y + 3xyz + y^3z + C_2. \tag{10}$$

Finally, consider the third position of $\mathbf{F}$, using (10), we get

$$f_z = 3xy + y^3 + (C_2)_z, \quad \Longrightarrow \quad C_2 = 0, \quad \text{(you can obtain any constant here)}$$

by comparing with the third position of $\mathbf{F}$. Thus, we have found a potential for $\mathbf{F}$ as

$$f(x, y, z) = x^2y + 3xyz + y^3z = \nabla \mathbf{F}.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (x^2y + 3xyz + y^3z)\Big|_{(0,1,-1)}^{(2,1,1)} = 11 - (-1) = 12.$$

6 (15.2), (15.4)

Consider the vector fields

$$\mathbf{F} = \Big((1+x)e^{x+y}, xe^{x+y} + 2y, -2z\Big),$$
$$\mathbf{G} = \Big((1+x)e^{x+y}, xe^{x+y} + 2z, -2y\Big).$$

a) Show that $\mathbf{F}$ is conservative by finding a potential for it.

b) Evaluate

$$\int_C \mathbf{G} \cdot d\mathbf{r},$$

where C is given by

$$\mathbf{r} = \Big((1-t)e^t, t, 2t\Big), \quad (0 \leq t \leq 1),$$

by taking advantage of the similarity between $\mathbf{F}$ and $\mathbf{G}$.

Solution.

a)

Observing the first position of $\mathbf{F}$, we have

$$\begin{aligned}\int (1+x)e^{x+y}\,dx = \int (1+x)\,d(e^{x+y}) =& (1+x)e^{x+y} - \int e^{x+y}dx \\ =& (1+x)e^{x+y} - e^{x+y} \\ =& xe^{x+y}.\end{aligned}$$

We can suppose the potential function as

$$f(x,y,z) = xe^{x+y} + C_1, \tag{11}$$

where C_1 is a function that depends on the variables y and z. So we now know that

$$f_x = (1+x)e^{x+y}.$$

Let us continue to consider the second position of $\mathbf{F}$. From our result of f in (11), we have

$$f_y = xe^{x+y} + (C_1)_y.$$

Comparing this result with the second position of $\mathbf{F}$, we get

$$(C_1)_y = 2y, \quad \Longrightarrow \quad C_1 = y^2 + C_2,$$

where C_2 is a function that depends on the variable z. So by now we have

$$f(x,y,z) = xe^{x+y} + y^2 + C_2. \tag{12}$$

Finally, consider the third position of $\mathbf{F}$, using (12), we get

$$f_z = (C_2)_z = -2z, \quad \Longrightarrow \quad C_2 = -z^2. \quad \text{(you can add any constant here)}$$

Thus, we have found a potential for $\mathbf{F}$ as

$$f(x, y, z) = xe^{x+y} + y^2 - z^2 = \nabla\mathbf{F},$$

which means $\mathbf{F}$ is conservative.

b)

From the expression of G, we

$$\begin{aligned}\mathbf{G} =&\Big((1+x)e^{x+y}, xe^{x+y} + 2z, -2y\Big)\\ =&\Big((1+x)e^{x+y}, xe^{x+y} + 2y, -2z\Big) + \Big(0, 2(z-y), 2(z-y)\Big)\\ =&\mathbf{F} + \Big(0, 2(z-y), 2(z-y)\Big).\end{aligned}$$

From the expression of C, we have

$$\mathbf{r}(0) = (1, 0, 0), \quad \mathbf{r}(1) = (0, 1, 2).$$

Thus,

$$\begin{aligned}\int_C \mathbf{G}\cdot\, d\mathbf{r} =& \int_C \mathbf{F}\cdot\, d\mathbf{r} + \int_C \Big(0, 2(z-y), 2(z-y)\Big)\cdot\, d\mathbf{r}\\ =&(xe^{x+y} + y^2 - z^2)\Big|_{(1,0,0)}^{(0,1,2)} + \int_0^1 \Big(0, 2(2t-t), 2(2t-t)\Big)\cdot \frac{d\mathbf{r}}{dt}\, dt\\ =&-3 - e + \int_0^1 \Big(0, 2(2t-t), 2(2t-t)\Big)\cdot(-te^t, 1, 2)\, dt\\ =&-3 - e + \int_0^1 6t\, dt\\ =&-3 - e + 3t^2\Big|_0^1\\ =&-e.\end{aligned}$$