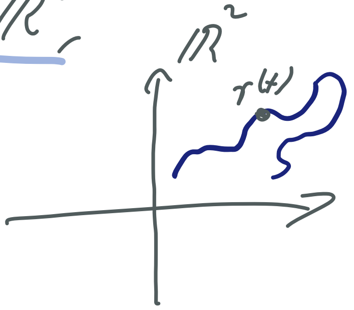
8.2 Kurver i planet

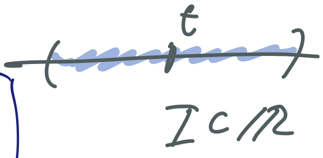
$\subset \quad \subseteq \qquad \subsetneq$

Def. En (parametrisk) kurve i planet er en kontinuerlig $\gamma : I \to \mathbb{R}^2$,

$I \subset \mathbb{R}$ er et intervall,

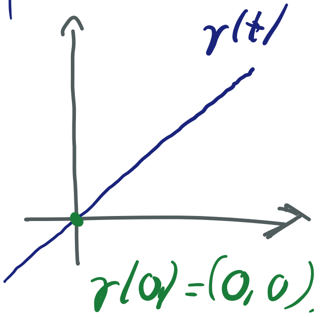
$$t \longmapsto \gamma(t) = (\gamma_1(t), \gamma_2(t)) = (x(t), y(t)).$$



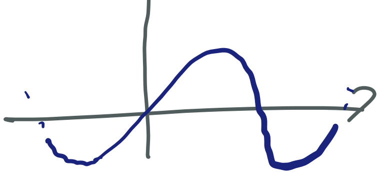


Obs! kurve kan bety så vel funksjonen $t \mapsto \gamma(t)$, som bildet $\{\gamma(t) : t \in I\}$

Eks. (i) $\gamma(t) = (x(t), y(t)) = (t, t)$, $t \in \mathbb{R}$.



(ii) $\gamma(t) = (t, \sin(t))$, $t \in \mathbb{R}$



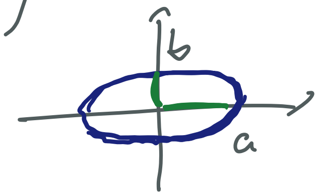
(i) og (ii) er grafer: $\{(x, f(x)) : x \in D_f\}$

Men:

(iii) $\gamma(t) = (a\sin(t), b\cos(t))$, $t \in [0, 2\pi)$

er det ikke $\boxed{\text{ellipse}}$

$$\frac{(x(t))^2}{a^2} + \frac{(y(t))^2}{b^2} = 1$$

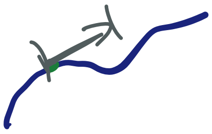


8.3 Glatte kurver (obs. skiller seg fra AE!)

Def. $\gamma : I \to \mathbb{R}^2$ er glatt dersom γ_1, γ_2 er kont. deriverbare (dvs $\dot{\gamma}$ er kont.)

med $|\dot{\gamma}(t)| = \sqrt{(\dot{\gamma}_1(t))^2 + (\dot{\gamma}_2(t))^2} \neq 0$ overalt på I.

farten ikke null, og $\dot{\gamma} = (\dot{\gamma}_1, \dot{\gamma}_2) = (\dot{x}, \dot{y})$ hastigheten eksisterer.



Def.

Helningen $\frac{dy}{dx}$ av en glatt kurve gis ved

$$\boxed{\frac{dy}{dx} \underset{\text{kj.reg.}}{=} \frac{dy/dt}{dx/dt} = \frac{\dot{y}(t)}{\dot{x}(t)} = \frac{\dot{\gamma}_2(t)}{\dot{\gamma}_1(t)}.}$$

Obs. krever $\frac{dx}{dt} \neq 0$.

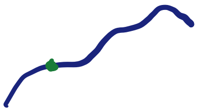
Alt. med kjerneregl. og omvendt funksjoner:

$\dot{x}(t) \neq 0 \iff \frac{dx}{dt} \neq 0 \implies x$ omvendbar

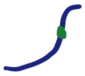
$\implies t = t(x) \iff \underbrace{\exists\, \gamma_1^{-1}}_{\text{'eksisterer'}} : x \mapsto t(x)$

Så: $\frac{dy}{dx} = \frac{d}{dx} y(t) = \frac{d}{dx} y(t(x)) = \frac{d}{dx} y(\underbrace{\gamma_1^{-1}(x)}_{t})$

$$\underset{\substack{\uparrow \\ \text{kj. reg.}}}{=} \dot{y}(t) \cdot \frac{1}{\gamma_1'(t)} = \frac{\dot{y}(t)}{\dot{x}(t)}.$$



Merk: Dersom $\dot{y}(t) \neq 0$, kan vi i stedet beregne $\frac{dx}{dy} = \frac{\dot{x}(t)}{\dot{y}(t)}$.



På samme måte: $\frac{d}{dx} y(t(x)) = \frac{d}{dx}\left(\frac{d}{dx} y(t(x))\right)$

$$= \frac{d}{dx}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right) \underset{\substack{\uparrow \\ \text{kj. reg.}}}{\overset{|\dot{x}| \neq 0}{=}} \underbrace{\frac{dt}{dx}\left(\frac{d}{dt}\right.}_{\frac{d}{dx}} \left.\frac{\dot{y}(t)}{\dot{x}(t)}\right)$$

$$= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^2} \frac{1}{\frac{dx}{dt}} = \boxed{\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}.}$$

8.4 Buelengde og areal

Vet: lengden av grafen $(x, y(x))$ fra $x=a$ til $x=b$.

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Så hvis $\dot{x} = \frac{dx}{dt} > 0$, gjelder:

$$\boxed{\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,\underbrace{dx}_{\frac{dx}{dt}dt}} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{\cancel{dx}}\right)^2\left(\frac{\cancel{dx}}{dt}\right)^2}\,dt$$

$$= \sqrt{\dot{x}^2+\dot{y}^2}\,dt = \boxed{|\dot{r}|\,dt.}$$

Obs. liknende utregning dersom $\frac{dy}{dt} \neq 0$.

Motiverer:

Def.
- Buelengden $L(r) = \int_a^b |\dot{r}(t)|\,dt$ er lengden av kurven r fra $t=a$ til $t=b$.
- Buelengdeparameteren $s(t) = \int_{t_0}^{t} |\dot{r}(t)|\,dt$ er lengden av r fra $t=t_0$ til t (variabel)
- Buelengdeelementet er $ds = |\dot{r}(t)|\,dt$.

Ettersom $\frac{ds}{dt} \underset{\substack{\uparrow \\ \text{fund.teo.}}}{=} |\dot{\gamma}(t)| \neq 0$ for glatte kurver,

$\left(s \xleftrightarrow{\text{omvendbar}} t\right)$

kan vi alltid skifte mellom t og s.

$$\boxed{\text{Hvis } |\dot{\gamma}| = 1 \text{ er } \gamma = \gamma(s) \text{ er parametrisert ved buelengde og } L(\gamma)\Big|_0^s = \int_0^s \underbrace{|\dot{\gamma}(u)|}_{1}\,du = s.}$$

Eks. Kurven $t \longmapsto \gamma(t) = (t^3 - 3t, t^2)$.

$t \in [-2, 2]$.

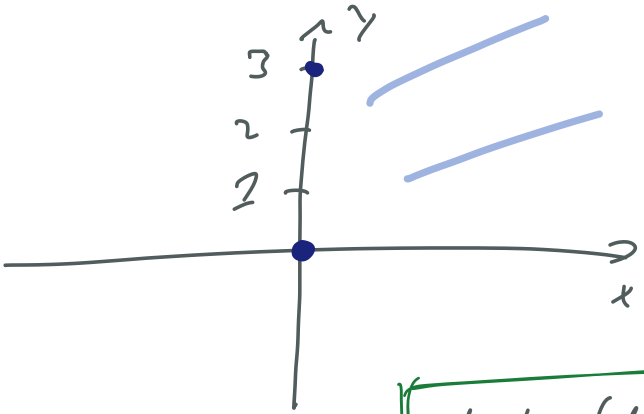
Fart: $\exists\ \dot{\gamma}(t) = \overbrace{(3t^2 - 3, 2t)}^{\text{hastigheten}}$ kont, og

$$\underbrace{|\dot{\gamma}(t)|}_{\text{farten}} = \Big(\big(3\underbrace{(t^2-1)}_{=0 \iff t = \pm 1}\big)^2 + \underbrace{(2t)}_{=0 \iff t=0}^2 \Big)^{\frac{1}{2}} \neq 0 \quad \forall t.$$

Så γ er glatt.

Tegne: $x = 0 \iff t^3 - 3t = 0 \iff t(t^2-3) = 0$

$$\iff \begin{cases} t = 0 & \Rightarrow y = 0 \\ \text{eller} & \\ t = \pm\sqrt{3} & \Rightarrow y = 3. \end{cases}$$



$$\boxed{\gamma(t) = (t^3-3t,\ t^2).}$$

Obs! (i) $y = t^2 \geq 0 \quad \forall t$

(ii) $\gamma(-t) = (-x(t), y(t))$

γ symmetrisk kring y-aksen.

$\longrightarrow$ Tilstrekkelig å studere $x, y \geq 0$.

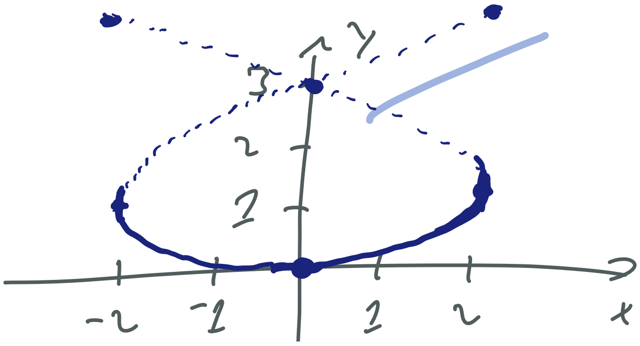
(alt. studer $t \geq 0$) ikke likt $x \geq 0$!

$\frac{dy}{dt} \Big/ \frac{dx}{dt}$

$$\frac{dy}{dx} = \frac{\dot{\gamma}_2(t)}{\dot{\gamma}_1(t)} = \frac{2}{3}\frac{t}{t^2-1}$$

$\gamma'(t)$	$-$	∞	$+$	0	$-$	∞	$+$		
t	-2		-1		0		1		2
γ	$(-2,4)$		$(2,1)$		$(0,0)$		$(-2,1)$		$(2,4)$

$$\boxed{\gamma(t) = (t^3-3t,\ t^2).}$$

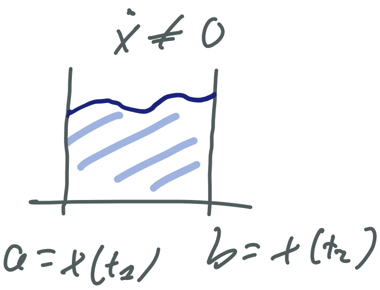


Buelengde: $s(t) = \int_0^t |\dot{r}(\tau)|\, d\tau = \dots$

$$= \int_0^t \sqrt{13\tau^4 - 18\tau^2 + 9}\, d\tau = \dots$$

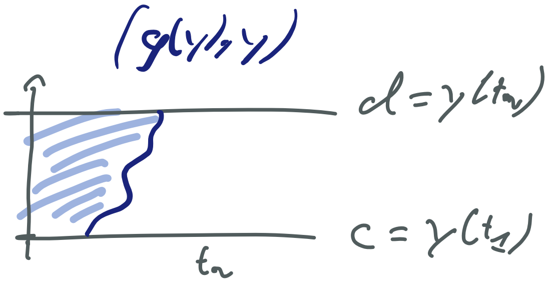
Areal: Sammenlikn med $(x, f(x))$
$(x(t), y(t))$

når $\dot{x} \neq 0$: $y(t) = y(t(x)) = f(x)$



$$A = \int_a^b f(x)\,dx \underset{\substack{\uparrow \\ \text{var. subst.} \\ x \leftrightarrow t}}{=} \int_{t_1}^{t_2} f(x(t)) \frac{dx}{dt}\, dt$$

$$= \boxed{\int_{t_1}^{t_2} y(t)\,\dot{x}(t)\,dt}$$

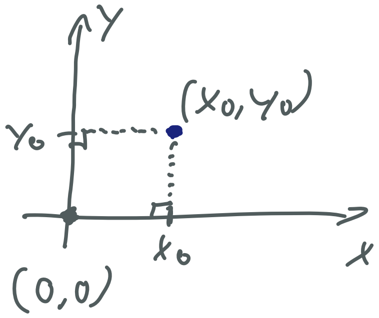


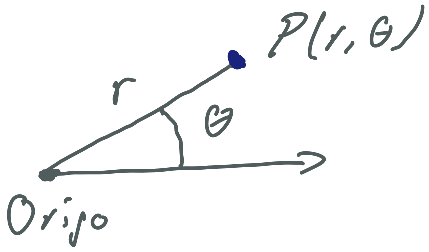
og for $\dot{y} \neq 0$:

$$A = \int_c^d g(y)\,dy \underset{\substack{\uparrow \\ \frac{dy}{dt} \neq 0}}{=} \int_{t_1}^{t_2} g(y(t)) \frac{dy}{dt}\, dt = \int_{t_2}^{t_2} x(t)\,\dot{y}(t)\,dt.$$

8.5-8.6 Polarkoordinater og kurver i polarkoord.

Kartesiske koord. | Polarkoord.

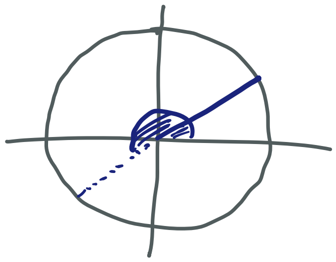




$$x = r\cos(\theta) \qquad\qquad \begin{cases} r^2 = x^2+y^2 \\ \tan(\theta) = \frac{y}{x} \end{cases}$$
$$y = r\sin(\theta) \quad \xLeftrightarrow[x\neq 0]{}$$

$0 \le \theta < 2\pi$, $r \geq 0$ gir entydighet

utenom for $r = 0$:



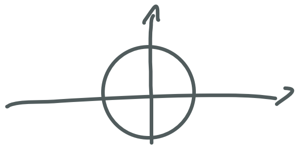
$P(0,\theta_1) = P(0,\theta_2)$

$\forall\, \theta_1, \theta_2$.

Noen vanlige geometriske likninger

(i) $r = r_0 \iff x^2+y^2 = r_0^2$

$(\theta \in \mathbb{R})$

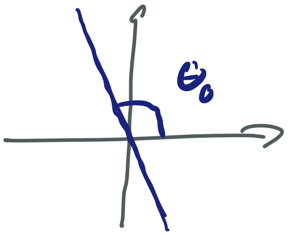


sirkel med radius r_0.

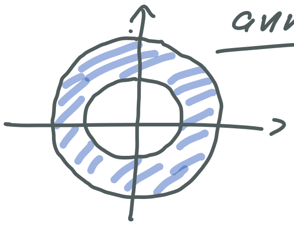
(ii) $\theta = \theta_0 \iff \frac{y}{x} = \tan(\theta_0) = \text{konst.}$

$(r \in \mathbb{R})$

linje med helning $\tan(\theta_0)$.



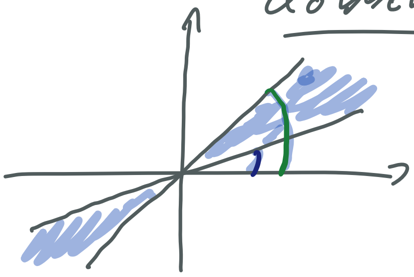
(iii) $\begin{cases} r_1 \le r \le r_2 \\ 0 \le \theta < 2\pi \end{cases}$



annulus

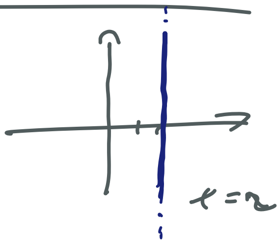
(iv) $\theta_1 \le \theta \le \theta_2$

$(r \in \mathbb{R})$



dobbeltkjegle

Eks. (i) $r\cos(\theta) = 2 \iff x = 2$

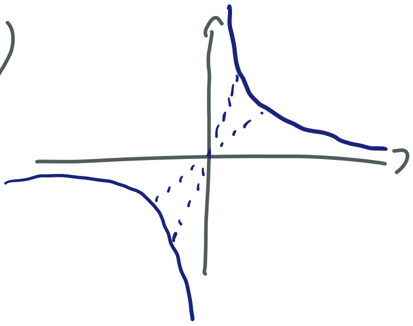


(ii) $r^2\cos(\theta)\sin(\theta) = 4$

$\iff xy = 4 \quad (y = \frac{4}{x}, x = \frac{4}{y})$

invariant $r \mapsto -r$

$\Rightarrow$ symmetri kring origo.



(iii) $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1 \iff \boxed{x^2 - y^2 = 1}$

$(x-y)(x+y) = 1$

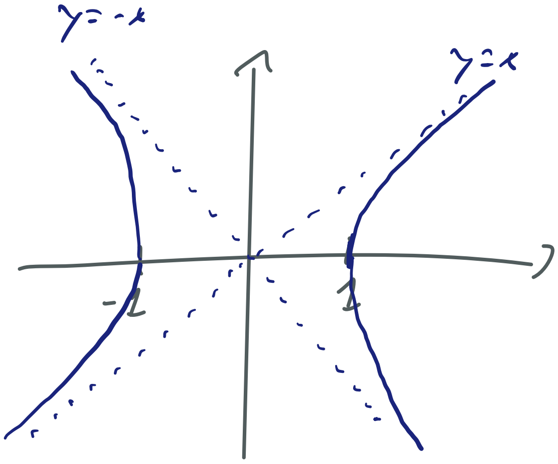
hyperbel

Symmetrier:

$x \mapsto -x$

$y \mapsto -y$

$r \mapsto -r$



$$\boxed{\begin{aligned} u &= x - y \\ v &= x + y \end{aligned}}$$

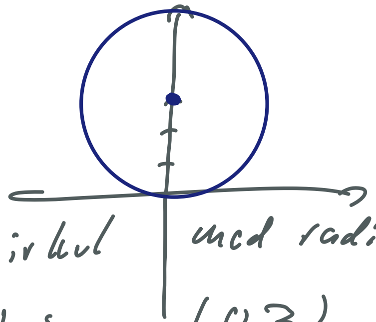
(iv) $x^2 + (y-3)^2 = 9$

$\iff x^2 + y^2 - 6y + \cancel{9} = \cancel{9}$

$\iff r^2 = 6r\sin\theta$

$\iff \underline{r = 6\sin\theta}$

$r = 0$ inkludert i $\theta = 0$.



sirkel med radius 3 kring $(0,3)$.

En 'polarkurve' er en kurve $r = f(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, dvs $\underset{(r\,,\,\theta)}{(f(\theta), \theta)}$ i polarkoord.

Denne er glatt dersom $f \in \underset{\text{kont. der.}}{C^1(\mathbb{R},\mathbb{R})}$.

Hellningen $\frac{dy}{dx}$:

$x = r\cos(\theta) = f(\theta)\cos(\theta)$

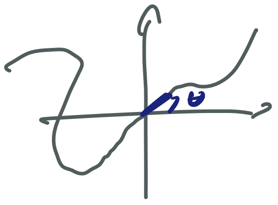
$y = r\sin(\theta) = f(\theta)\sin(\theta)$

$$\boxed{\begin{aligned}\frac{dx}{d\theta} &= f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\\ \frac{dy}{d\theta} &= f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\end{aligned}}$$

Så: $$\frac{dy}{dx} \underset{\underset{\text{gitt } dx/d\theta \neq 0}{\uparrow}}{=} \frac{dy/d\theta}{dx/d\theta} = \boxed{\frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}}$$

Spesialtilfelle : kurven passerer origo : $f(\theta_0) = 0$.

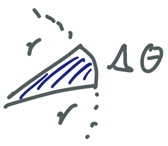
$$\boxed{\frac{dy}{dx} = \Big|_{r = f(\theta_0) = 0} \tan(\theta_0).}$$



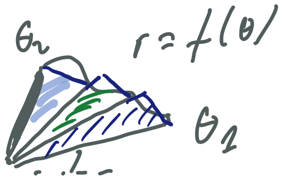
Hellning er vinkel.

Areal 'innenfor' polarkurver

Areal av et sirkelsegment:



$$\boxed{\underset{\underset{\text{sirkel}}{\uparrow}}{\pi r^2}\;\underset{\underset{\text{del av}}{\uparrow}}{\frac{\Delta\theta}{2\pi}}}$$



sirkelen

Approksimasjon: $\sum_i \frac{r_i^2}{2}\Delta\theta_i$

$$\rightsquigarrow \boxed{\int_{\theta_1}^{\theta_2} \frac{f(\theta)^2}{2}\, d\theta}$$

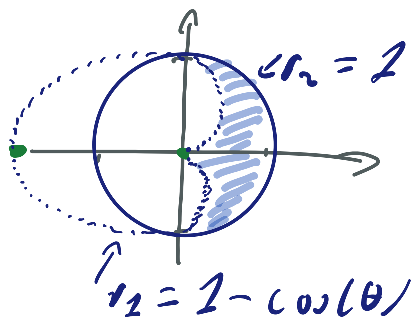
total areal

Eks. Areal mellom $\{r=1\}$ og $\{r=1-\cos(\theta)\}$

sirkel kardiode

$$r_2 \geq r_1 \iff 1 \geq 1-\cos(\theta)$$
$$\iff \cos(\theta) \geq 0$$
$$\iff -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$[-\pi, \pi]$



$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[1^2 - (1-\cos(\theta))^2\right] d\theta = \boxed{\int r_2^2 - \int r_1^2}$$

$$2\cos(\theta) - \underbrace{\cos^2(\theta)}_{\frac{1+\cos(2\theta)}{2}}$$

sym.

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^{\pi/2}\left(2\cos(\theta) - \frac{1}{2} - \frac{\cos(2\theta)}{2}\right)d\theta$$

$$= 2\sin(\theta)\Big|_0^{\pi/2} - \frac{\pi}{4} - \frac{\sin(2\theta)}{4}\Big|_0^{\pi/2}$$

$$= 2 - \frac{\pi}{4}.$$

Buelengde for kurve $r = f(\theta)$

$x = r\cos(\theta) = f(\theta)\cos(\theta)$

$y = r\sin(\theta) = f(\theta)\sin(\theta)$

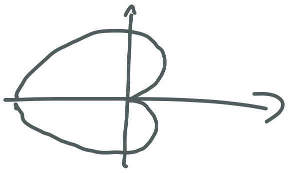
$$\Rightarrow \boxed{L_\gamma = \int_{\theta_1}^{\theta_2} \underbrace{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)^{\frac{1}{2}}}_{|(\dot{x},\dot{y})| = |\dot{\gamma}|} d\theta}$$

$$\left(\frac{dx}{d\theta}\right)^2 = \left(\underline{f'(\theta)\cos(\theta)} - \underline{f(\theta)\sin(\theta)}\right)^2 \quad \cos^2 + \sin^2 = 1 \longrightarrow \underline{(f'(\theta))^2} + \underline{f(\theta))^2}$$

$$\left(\frac{dy}{d\theta}\right)^2 = \left(\underline{f'(\theta)\sin\theta} + \underline{f(\theta)\cos(\theta)}\right)^2$$

$$\Rightarrow \boxed{L_\gamma = \int_{\theta_1}^{\theta_2} \Big[\underbrace{(f(\theta))^2}_{r(\theta)} + \underbrace{(f'(\theta))^2}_{r'(\theta)}\Big]^{\frac{1}{2}} d\theta}$$

Eks. Buelengde av kardioden $\{r = 1 - \cos(\theta)\}$ for $0 \leq \theta \leq 2\pi$.



$$\begin{cases} r(\theta) = 1 - \cos(\theta) \\ \dfrac{dr}{d\theta} = \sin(\theta) \end{cases} \Rightarrow r^2 + (r')^2 = (1 - \cos(\theta))^2 + \sin^2(\theta)$$

$$= 1 - 2\cos(\theta) + \underbrace{\cos^2(\theta) + \sin^2(\theta)}_{1}$$

$$= 2\,\underline{(1 - \cos(\theta))}$$

$$L_r = \int_0^{2\pi} \left(r^2(\theta) + (r'(\theta))^2\right)^{\frac{1}{2}} d\theta = 2\int_0^{\pi} \sqrt{2\underbrace{(1-\cos(\theta))}_{2\sin^2(\frac{\theta}{2})}}\, d\theta$$

$$= 4\int_0^{\pi} \sin\left(\tfrac{\theta}{2}\right) d\theta = 4\left[-2\cos\left(\tfrac{\theta}{2}\right)\right]_0^{\pi} = 8.$$

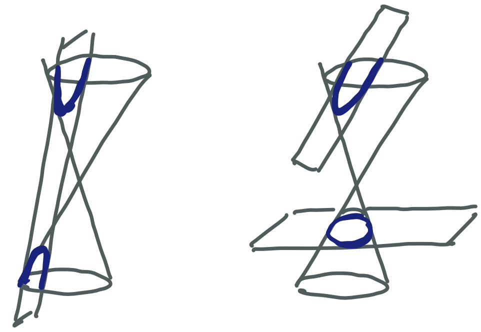
$\uparrow$

$\sin(\frac{\theta}{2}) \geq 0$

for $0 \leq \theta \leq \pi$.

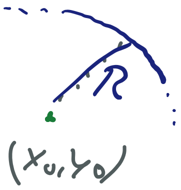
8.1 Kjeglesnitt

La d betegne avstand i (x,y)-planet.



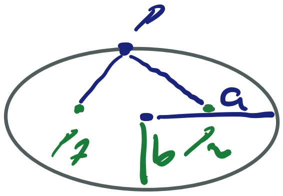
Sirkel: $\{P : d(P, P_0) = \text{konst}\}$

$$(x-x_0)^2 + (y-y_0)^2 = R^2$$

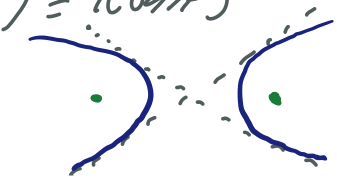


Ellipse: $\{P : d(P, P_1) + d(P, P_2) = \text{konst}\}$

$$\left(\frac{x-x_0}{a}\right)^2 + \left(\frac{y-y_0}{b}\right)^2 = 1$$



Hyperbel $\{P : d(P, P_1) - d(P, P_2) = \text{konst}\}$

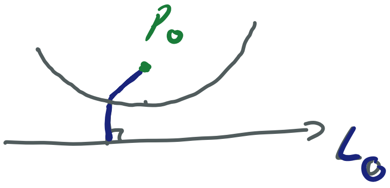


$$\left(\frac{x-x_0}{a}\right)^2-\left(\frac{y-y_0}{b}\right)^2=1$$

Asymptoter: $\left|\frac{x}{a}\right|=\left|\frac{y}{b}\right|$

Parabel $\{P : d(P,P_0)=d(P,L_0)\}$

$$y-y_0=\underset{\substack{\uparrow\\ c\neq 0}}{c}\,(x-x_0)^2$$

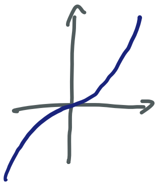


Vanlige parameterisenger:

sirkel/ellipse: $\gamma(t)=(\underset{x_0+}{\wedge}a\cos(t),\ \underset{y_0+}{\wedge}b\sin(t)),\quad t\in[0,2\pi)$

hyperbel: $\gamma(t)=(a\cosh(t),\ b\sinh(t)),\ t\in\mathbb{R}$

$\cosh(t)=\frac{e^t+e^{-t}}{2}$, $\sinh(t)=\frac{e^t-e^{-t}}{2}$



obs! $\cos^2(t)+\sin^2(t)=1$ $\quad\cosh^2(t)-\sinh^2(t)=1$

parabel: $\gamma(t)=(t,t^2),\ t\in\mathbb{R}$

(t^2,t)

11.1 og 11.3 Funksjoner $I \subset \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{Z}_{\geq 1}$

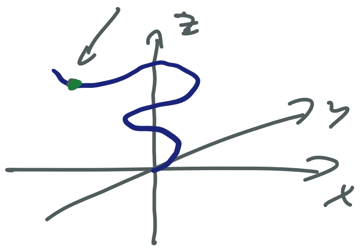
(kurver, vektorvaluerte funksjoner)

$$I \ni t \longmapsto \underbrace{\gamma(t)}_{\mathbb{R}^n} = (\gamma_1(t), \underset{\underset{\mathbb{R}}{\uparrow}}{\gamma_2(t)}, \dots, \gamma_n(t))$$

$(x(t), y(t), z(t))$

Eks. posisjon av partikkel

$\gamma(t) = (x(t), y(t), z(t))$



I stedet for $(x(t), y(t), z(t))$ kan vi skrive

$$\gamma(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$\mathbf{i} = (1,0,0)$ $\quad \mathbf{j} = (0,1,0)$ $\quad \mathbf{k} = (0,0,1)$

e_1 $\quad e_2$ $\quad e_3$

Def. $\gamma : I \to \mathbb{R}^n$ kont i t_0 $\overset{\text{def.}}{\iff}$

$\forall \varepsilon > 0 \ \exists \delta > 0;$

$$|t - t_0| < \delta \implies |\underbrace{\gamma(t) - \gamma(t_0)}_{\mathbb{R}^n}| < \varepsilon.$$

Obs! (i) $\quad |\gamma(t) - \gamma(t_0)| = \left(\sum_{j=1}^{n} (\gamma_j(t) - \gamma_j(t_0))^2 \right)^{1/2}$

; $\mathbb{R}^3$: $\sqrt{(x(t)-x(t_0))^2 + (y(t)-y(t_0))^2 + (z(t)-z(t_0))^2}$

(ii) $$\max_j |\gamma_j(t)-\gamma(t_0)| \le |\gamma(t)-\gamma(t_0)|$$

$$\le \sum_{j=1}^{n} |\gamma_j(t)-\gamma_j(t_0)| \le n \max_j |\gamma_j(t)-\gamma_j(t_0)|$$

$\triangle$ $\sqrt{a^2+b^2} \le |a|+|b|$

$$\Longrightarrow |\gamma(t)-\gamma(t_0)| \sim \max_j |\gamma_j(t)-\gamma_j(t_0)|$$

$\Longrightarrow$ $\boxed{\gamma : I \to \mathbb{R}^n \text{ kont. dersom } \gamma_1, \gamma_2, \ldots, \gamma_n \text{ er kont } I \to \mathbb{R}.}$

Def. $\gamma : I \to \mathbb{R}^n$ deriverbar i $t_0 \overset{\text{def.}}{\Longleftrightarrow}$

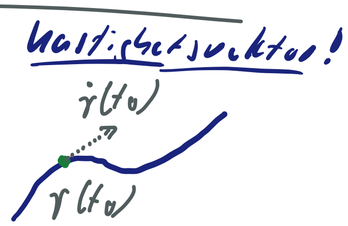
$$\exists\ \dot\gamma(t_0) = \lim_{t\to t_0} \frac{\overbrace{\gamma(t)-\gamma(t_0)}^{\mathbb{R}^n}}{\underbrace{t-t_0}_{\mathbb{R}}}$$

Som ovenfor, $\exists\ \dot\gamma(t) = (\dot\gamma_1(t), \dot\gamma_2(t), \ldots, \dot\gamma_n(t))$

$\Longleftrightarrow \exists\ \dot\gamma_j(t)\ \forall\ j = 1, \ldots, n.$

Hva 'er' $\dot\gamma(t_0)$?

$$\dot\gamma(t_0) = \lim_{h\to 0} \frac{\gamma(t_0+h)-\gamma(t_0)}{h}$$



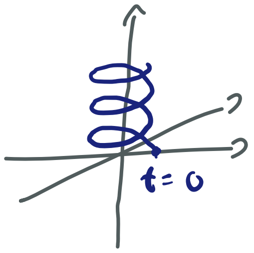
$|\dot{\gamma}(t)|$ fart (skalar): γ glatt $\Longleftrightarrow |\dot{\gamma}| > 0$ $\forall t \in I$.

$\ddot{\gamma}(t)$ akselerasjon vektor!

Ofte: $\dot{\gamma} = v$, $\ddot{\gamma} = a$. Merk: $v = \underbrace{\frac{v}{|v|}}_{\text{enhetstangent-vektor } T} |v|$

'velocity' 'acceleration'

Eks. Heliks $\gamma(t) = (\cos(t), \sin(t), t)$

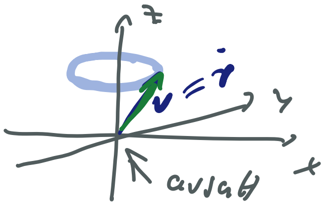


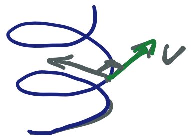
- γ kont? Ja, fordi: hver komponent kont (fundamentale funksj.)
- γ der.bar? Ja, fordi: $\dot{\gamma}(t) = \underset{V}{\boxed{(-\sin(t), \cos(t), 1)}}$
- γ glatt? $|\dot{\gamma}(t)| = \left(\underbrace{(-\sin(t))^2 + (\cos(t))^2}_{1} + 1^2\right)^{1/2}$

$$= \underbrace{\sqrt{2}}_{|v|} > 0 \quad \text{glatt}$$

- akselerasjon: $a = \ddot{\gamma}(t) = \dot{v}(t) = (-\cos(t), -\sin(t), 0)$

Merk:





avsatt i $\gamma(t)$

i origo

$$v \cdot a = (-\sin(t), \cos(t), 1) \cdot (-\cos(t), -\sin(t), 0)$$

$$= \cancel{\sin(t)\cos(t)} - \cancel{\cos(t)\sin(t)} + 0 = 0$$

□

Prop. $\gamma(t) : I \to \mathbb{R}^n$ deriverbar med $|\gamma(t)| = \text{konst} \ \forall t \in I \Rightarrow \dot\gamma(t) \cdot \gamma(t) = 0$ $\forall t.$

Obs! For helikser spiller $\dot\gamma$ rollen til γ i proposisjonen.

Bevis. $|\gamma(t)| = \text{konst.} \iff |\gamma(t)|^2 = \text{konst.}$

$$\iff \sum_{j=1}^{n} \gamma_j^2(t) = \text{konst}$$

$$\overset{d/dt}{\Longrightarrow} 2 \sum_{j=1}^{n} \underset{\overset{\uparrow}{\mathbb{R}}}{\gamma_j(t)}\, \underset{\nearrow}{\dot\gamma_j(t)} = 2\gamma(t) \cdot \dot\gamma(t) = 0$$

$$\iff \gamma(t) \cdot \dot\gamma(t) = 0 \iff \gamma \perp \dot\gamma \quad \forall t \in I.$$ □

Derivasjonsregler for kurver

$\mu, \lambda \in \mathbb{R}$, u, v deriverbare $I \to \mathbb{R}^n$

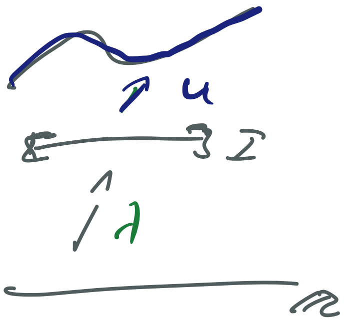
(i) $(\mu u+\lambda v)'=\mu u'+\lambda v'$

(ii) $(u\cdot v)'=u'\cdot v+u\cdot v'$

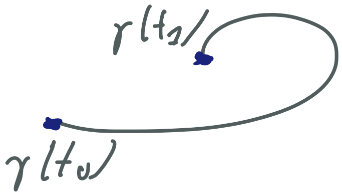
(iii) $(u\times v)'=u'\times v+u\times v'$,

og for $\lambda(t)$ funksjon $\mathbb{R}\to I$,

(iv) $u(\lambda(t))'=\underbrace{u'(\lambda(t))}_{\mathbb{R}^n}\underbrace{\lambda'(t)}_{\mathbb{R}}$.



Buelengde



Som tidligere:

Def. • En glatt kurve $\gamma:[t_0,t_1]\to\mathbb{R}^n$ har buelengde mellom $\gamma(t_0)$ til $\gamma(t_1)$:

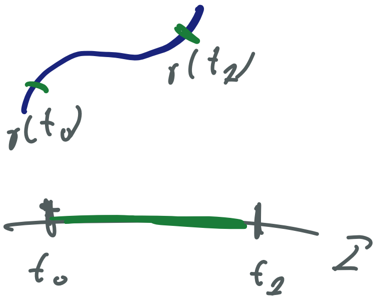
$$L(\gamma)=\int_{t_0}^{t_1}|\dot\gamma(t)|\,dt=\int_{t_0}^{t_1}\sqrt{\dot\gamma_1^2+\dots+\dot\gamma_n^2}\,dt$$

- $s(t)=\int_{t_0}^{t}|\dot\gamma(\tau)|\,d\tau$ er buelengdevariabelen.

- $ds=|\dot\gamma(t)|\,dt$ er buelengdeelementet.

- Dersom $|\dot\gamma(t)|=1\ \forall t\in I$, er γ er parametrisert ved buelengde.

og $s = t - t_0$.



Obs! $\frac{ds}{dt} = |\dot{\gamma}(t)| \neq 0$

$\Rightarrow s \xleftrightarrow[\text{inv. bar.}]{} t$

Eks. Helix $(\cos(t), \sin(t), t)$

$t \in [0, 2\pi]$

$$L(\gamma) = \int_0^{2\pi} |\dot{\gamma}(t)|\, dt \underset{\text{se fredag}}{=} \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{8}\,\pi.$$

$$s(t) = \int_0^t |\dot{\gamma}(\tau)|\, d\tau = \int_0^t \sqrt{2} = \sqrt{2}\, t$$

$$\Rightarrow \boxed{t = \frac{s}{\sqrt{2}}.}$$

Så $\gamma(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\right)$

er parametrisert ved sin egen lengde.

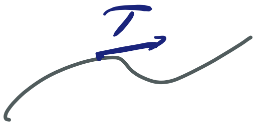
11.4 Krumning, torsjon og Frenetrammen.

Krumning forholder seg til fart, som den andrederiverte til den første der.

for vanlige funksjoner.

$\exists \; |\dot{\gamma}| \neq 0$

Husk: $\gamma : I \to \mathbb{R}^n$ glatt



$$T = \frac{v(t)}{|v(t)|} = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$$ enhetstangenten

Def. Krumningen $\kappa \overset{\text{def.}}{=} \left|\frac{dT}{ds}\right|$

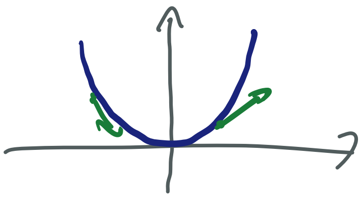
↖ bue lengde var.

Hva er $\frac{dT}{ds}$? Den deriverte til tangenten når hastigheten er konst 1 langs kurven.

Merk: $\frac{ds}{dt} = |\dot{\gamma}(t)| = |v(t)| \Rightarrow \frac{dt}{ds} = \frac{1}{|v(t)|}$

Så $$\boxed{\kappa = \left|\frac{dT}{ds}\right| = \left|\underbrace{\frac{dT}{dt}}_{\mathbb{R}^n} \underbrace{\frac{dt}{ds}}_{\mathbb{R}}\right| = \frac{1}{|v(t)|}\left|\frac{dT}{dt}\right|}$$

Eks. (i $\mathbb{R}^2$)



(i) parabel $\gamma(t) = (t, t^2)$, $t \in \mathbb{R}$

$\dot{\gamma}(t) = v(t) = (1, 2t)$, $|\dot{\gamma}(t)| = |v(t)| = \sqrt{1 + 4t^2} \geq 1$

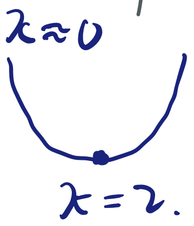
$$T = \frac{v}{|v|} = \frac{(1,2t)}{\sqrt{1+4t^2}}$$

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\right| \frac{1}{|v|}$$

$$\left|\frac{dT}{dt}\right| \underset{\uparrow \text{sjekk!}}{=} \left|\frac{(-4t,2)}{(1+4t^2)^{3/2}}\right| = \frac{\sqrt{4(4t^2)+4}}{(1+4t^2)^{3/2}} = \frac{2}{1+4t^2}.$$

Så: $\boxed{\kappa = \frac{2}{1+4t^2}\cdot\frac{1}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}.}$

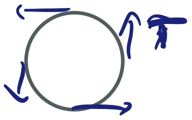
$\kappa \to 0$ da $t \to \pm\infty$



(ii) Sirkel: $r(\theta) = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi)$.

$$|\dot{r}(\theta)| = |(-\sin(\theta), \cos(\theta))| = 1.$$

Så $s = \theta$ er buelengden.



$$\Rightarrow T = \frac{v}{|v|} = v = (-\sin(\theta), \cos(\theta))$$

og $\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{d\theta}\right| = \left|\frac{dv}{d\theta}\right|$

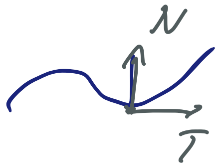
$= |(-\cos(\theta), -\sin(\theta))| = 1.$

Enhetssirkelen har konstant krumning 1.

Enhetsnormalen

Ettersom $|T(t)| = 1 \ \forall t$, følger

$T \cdot T' = 0$, dvs T' normal til T.



Def. Enhetsnormal

$$\boxed{N \overset{\text{def.}}{=} \frac{dT/ds}{|dT/ds|} = \frac{1}{\kappa}\frac{dT}{ds}}$$

Ved kjerneregelen

$\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt}$, så

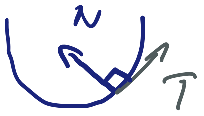
$$\frac{\frac{dT}{dt}\cancel{\frac{dt}{ds}}^{\,>0}}{\left|\frac{dT}{dt}\cancel{\frac{dt}{ds}}\right|}$$

$$\boxed{N = \frac{dT/dt}{|dT/dt|} \quad \text{gjelder osv.}}$$

κ er til N, som $|v|$ er til T.

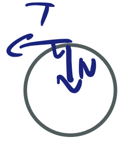
Eks. (i) parabelen $T = \frac{(1,2t)}{1+4t^2}$

$N = \frac{(-2t, 1)}{\sqrt{1+4t^2}}$ med lengde 1.



Obs! $T \cdot N = 0$.

(ii) sirkelen $T = (-\sin(\theta), \cos(\theta))$



$T \perp N$. $N = (-\cos(\theta), -\sin(\theta))$

Så i $\mathbb{R}^2$ er $\{T, N\}$ et lokalt (langs kurven) koordinatsystem / ortogonal basis.

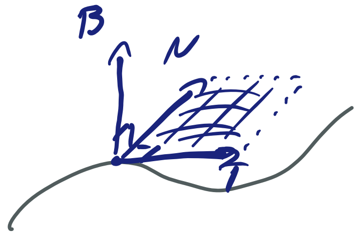
I $\mathbb{R}^3$ kan vi legge til

$$\boxed{B \overset{\text{def.}}{=} T \times N}$$

↑
Binormal

$B \perp T$, $B \perp N$,
$|B| = |T \times N| = 1$

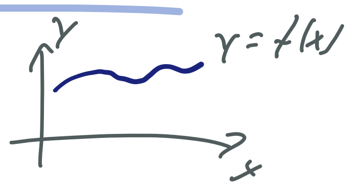
$$\boxed{\text{torsjon} \quad \frac{dB}{ds} = -\tau N}$$



Frenetrammen $\{T, N, B\}$ 3-dim. lokalt koord. system langs kurven.

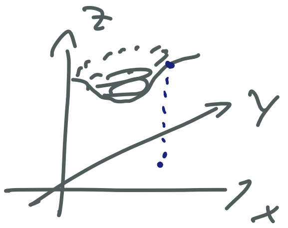
12.1 Funksjoner av flere variabler

Reell funksjon: $f: \mathbb{R} \to \mathbb{R}$



Kurve: $\gamma: \mathbb{R} \to \mathbb{R}^n$

Nå: $F: \mathbb{R}^n \to \mathbb{R}$



$z = F(x,y)$

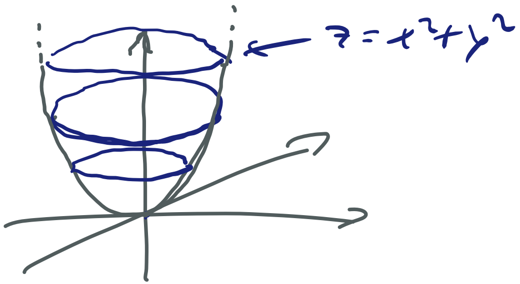
Enkelt eksempel:

$$\boxed{\begin{array}{l} F: \mathbb{R}^2 \to \mathbb{R} \\ (x,y) \mapsto x^2+y^2 \end{array}}$$

$F(x,y) = x^2+y^2$

Graf i $\mathbb{R}^3$:

$\{(x,y,z) : z = x^2+y^2\}$



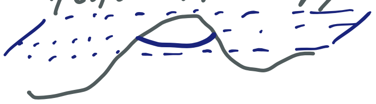
paraboloide
(se 10.5)

Grafen er en flate (ofte)

Generelt $\{F(x_1, \ldots, x_n) = c\}$ nivåmengde.

Nivåmengdene $\{F(x,y)=c\}$ er nivåkurver (oftest)

tenk fjelltopp



Def: En funksjon $F: U \subset \mathbb{R}^n \longrightarrow F(U) \subset \mathbb{R}$

$$x=(x_1,x_2,\dots,x_n) \longmapsto F(x)=F(x_1,\dots,x_n)$$

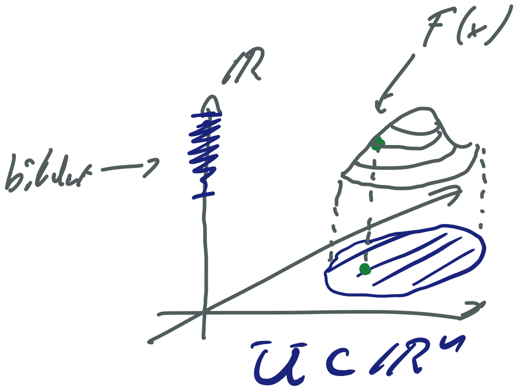
Har:

(i) Definisjonsmengde U

(domene)

(ii) kodomene $\mathbb{R}$

(iii) Et bilde $\{F(x): x \in U\} \subset \mathbb{R}$



Def. Dersom U ikke er gitt, er

$\{x \in \mathbb{R}^n : F(x) \in \mathbb{R}\}$ den naturlige

definisjonsmengden.

Eks. $$F(x_1,x_2)=\frac{x_1^2+x_2^2}{(x_1-1)^2}$$

Har naturlig def. mengde $\{(x_1,x_2): x_1 \neq 1\}$, og bilde $\mathbb{R}_{\geq 0} = [0, \infty)$.

Åpne og lukkede mengder

Def. Et punkt $x_0 \in \mathbb{R}^n$ er:

(i) et indre punkt i $U \subset \mathbb{R}^n$ dersom

$$\exists \varepsilon > 0 : \quad B_\varepsilon(x_0) \subset U$$

↑ ball med radius ε

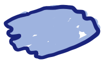
$\{x : |x - x_0| < \varepsilon\}$

(ii) et randpunkt til U dersom

$$\forall \varepsilon > 0 : \quad B_\varepsilon(x_0) \cap U \neq \emptyset$$

men $B_\varepsilon(x_0) \not\subset U$

Def. - Mengden av randpunkter til U kalles randen til U, ∂U.

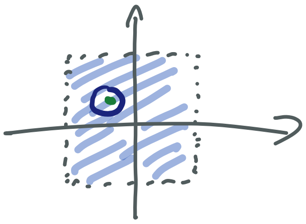


- $\overline{U} = U \cup \partial U$ kalles tillukningen av U.

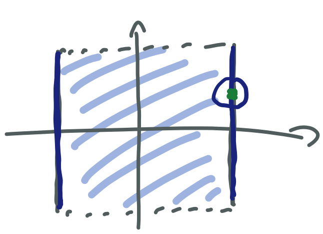
- Mengden U er åpen dersom alle punkter $x_0 \in U$ er indre punkter: $\partial U \cap U = \emptyset$.

- **lukket** dersom alle randpunkter tilhører U; $U = \overline{U}$.

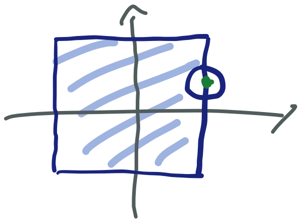
Eks. · $\{|x_1| < 1, |x_2| < 1\}$ åpen, ikke lukket.



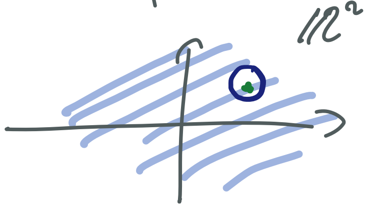
· $\{|x_1| \le 1, |x_2| < 1\}$ ikke åpen, ikke lukket.



· $\{|x_1| \le 1, |x_2| \le 1\}$ lukket, ikke åpen.



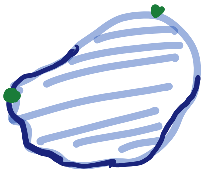
· $\mathbb{R}^2$ er åpen og lukket. (finnes ingen randpunkter)



Grenseverdier og kontinuitet

Def. $F(x)$ har grenseverdi $L \in \mathbb{R}$ i $x_0 \in \overline{U}$,

$$\lim_{x \to x_0} F(x) = L, \text{ dersom } \forall \varepsilon > 0 \; \exists \delta > 0;$$



$$|\underbrace{F(x) - L}_{\mathbb{R}}| < \varepsilon \text{ når } 0 < |\underbrace{x - x_0}_{\mathbb{R}^n}| < \delta \text{ og } x \in U.$$

obs! x_0 trenger ikke tilhøre U.

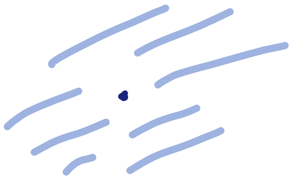
(ii) F er kont. i $x_0 \in U$ dersom

$$\exists \lim_{x \to x_0} F(x) \overset{\uparrow}{=} F(x_0)$$

x_0 tilhører U!

Eks. $F : U = \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$

$$(x_1, x_2) \longmapsto \frac{x_1^4 - x_2^4}{x_1^2 + x_2^2}$$



Grenseverdi i $(0,0)$?

$$F(x_1,x_2) = \frac{\cancel{(x_1^2 + x_2^2)}(x_1^2 - x_2^2)}{\cancel{x_1^2 + x_2^2}} = x_1^2 - x_2^2$$

$\forall (x_1, x_2) \in U$

L $\quad$ Δ

Så $|F(x_1,x_2) - 0| = \underbrace{|x_1^2 - x_2^2|}_{\text{vil } < \varepsilon} \leq |x_1^2| + |x_2^2|$

$= x_1^2 + x_2^2 < \varepsilon$

når $|(x_1,x_2) - (0,0)| = (x_1^2 + x_2^2)^{1/2} < \delta$.

dersom $\delta \leq \sqrt{\varepsilon}$. $\Longrightarrow \lim_{(x_1,x_2) \to (0,0)} F(x_1,x_2) = 0$.

Teorem Summer og produkter av kont. funk. er kont.

(for summer)

Bevis La $F, G : U \subset \mathbb{R}^n \to \mathbb{R}$ kont.

La $x_0 \in U$, $\varepsilon > 0$.

Betrakt $|(F+G)(x) - (F+G)(x_0)|$

$$= |F(x) + G(x) - F(x_0) - G(x_0)|$$

$$\underset{\Delta}{\leq} \underbrace{|F(x) - F(x_0)|}_{\leq \frac{\varepsilon}{2}} + \underbrace{|G(x) - G(x_0)|}_{\varepsilon/2} =: (*)$$

F, G kont $\Longrightarrow \exists\, \delta_F$ og δ_G og $\delta = \min\{\delta_F, \delta_G\}$;

$(*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $|x - x_0| < \delta$ og $x \in U$.

$\Longrightarrow$ $F+G$ kont i x_0 $\xRightarrow{x_0 \text{ vilk.}}$ $F+G$ på U. $\square$

Eks. Vet $x_1 \mapsto x_1^2$ kont $\mathbb{R} \to \mathbb{R}$

$\underset{\text{def av kont.}}{\Longrightarrow}$ $(x_1, x_2) \mapsto x_1^2$ kont $\mathbb{R}^2 \to \mathbb{R}$

$\underset{\text{teorem}}{\Longrightarrow}$ $\boxed{x_1^2 - x_2^2}$ kont $\mathbb{R}^2 \to \mathbb{R}$

$\Longrightarrow$ $\dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$ kont på $\mathbb{R}^2 \setminus \{(0,0)\}$.

Teorem

12.3 Partielt derivert

For funksjoner $f : \mathbb{R} \to \mathbb{R}$ er

(i) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, og

(ii) $f(x+h) = f(x) + f'(x)h + h\underbrace{\varepsilon(h)}_{\to 0 \text{ da } h \to 0}$

ekvivalente def. av den deriverte.

I høyere dim. gir de to forskjellige deriverte.

12.4 Høyere deriverte

Eks. $\partial_{x_1}\partial_{x_2}(x_1^2 e^{2x_2}) = \partial_{x_1}(2x_1^2 e^{2x_2}) = 4x_1 e^{2x_2}$.

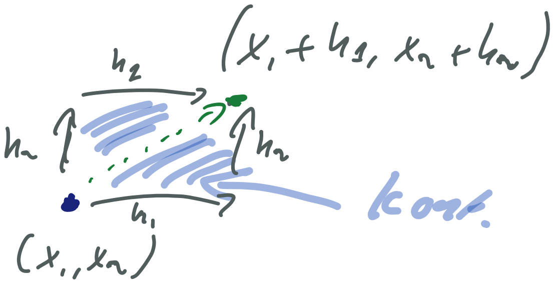
Teorem (Schwarz)

$$\partial_{x_1}\partial_{x_2} F = \partial_{x_2}\partial_{x_1} F$$

dersom $\partial_{x_1}^2 F$, $\partial_{x_1}\partial_{x_2} F$, $\partial_{x_2}\partial_{x_1} F$, $\partial_{x_2}^2 F$ er kont.

Uten bevis.

Idé



To 'moteksempler'

(i) $\exists\, \partial_v F$ men $F \notin C^1$.

(ii) $\exists\, \partial_{x_1}\partial_{x_2} F \neq \partial_{x_2}\partial_{x_1} F$

(i) $F : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \longmapsto \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Merk: $\underset{r\neq 0}{F(x,y)} = \frac{\cancel{2}r^{\cancel{2}}\cos(\theta)\sin(\theta)}{\cancel{2r}} = \boxed{\frac{r}{2}}\underbrace{\sin(2\theta)}_{|\cdot|\le 1} \xrightarrow{r\to 0} 0$

F kont på hele $\mathbb{R}^2$.

La $v = (\cos(\alpha), \sin(\alpha))$, $|v| = 1$. Fikser α,

$$\partial_v F(0,0) \overset{\text{def.}}{=} \lim_{t\to 0} \frac{F(tv) - \cancel{F(0,0)}}{t} = \frac{\cancel{\frac{t}{2}}\sin(2\alpha) \cancel{-0}}{\cancel{t}}$$

$\lim_{t\to 0}$

$$= \frac{1}{2}\sin(2\alpha).$$

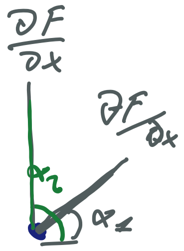
$$\Longrightarrow \quad \exists\, \partial_v F(0,0) \quad \forall \alpha \in [0, 2\pi).$$

Men: $\frac{\partial F}{\partial x}(x,y) = \frac{\overset{r}{y}}{\underset{r}{\sqrt{x^2+y^2}}} - \frac{\overset{r^3}{\cancel{2}x^2y}}{\underset{r^3}{\cancel{2}(x^2+y^2)^{3/2}}} =$

$(x,y) \neq (0,0)$

$$= \underline{\sin(\theta)[1 - \cos^2(\theta)]}.$$

og $\frac{\partial F}{\partial y}(x,y) = \underline{\cos(\theta)[1 - \sin^2(\theta)]}$



$\Longrightarrow \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$ kont på $\mathbb{R}^2 \setminus \{(0,0)\}$,

men mangler grenseverdi i $(0,0)$.

$F \notin C^1(\mathbb{R}^2, \mathbb{R})$

(+ kan vise: F ikke deriverbar i origo)

(ii) $$F(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

- $$F(x,y) = \frac{2r^2 \cos(\theta)\sin(\theta)\cancel{r^2}(\cos^2(\theta) - \sin^2(\theta))}{2\cancel{r^2}}$$

$$= \frac{r^2}{2\cdot 2} 2\sin(2\theta)\cos(2\theta) = \underbrace{\frac{r^2}{4}\underbrace{\sin(4\theta)}_{|\cdot| \leq 1} \xrightarrow{r \to 0} 0}_{F \text{ kont på } \mathbb{R}^2}$$

- Også C^1 (sjekk).

blir lagt ut i notaten

Men $$\partial_y(\partial_x F)(0,0) = \cdots = \partial_y(-y) = -1.$$
$$\neq$$
$$\partial_x(\partial_y F)(0,0) = \cdots = \partial_x(x) = 1.$$

Årsak: 'To derivate i origo eliminerer r^2, og θ blir viktig.'

For den interesserte, beregning av andrederiverte i origo for

$F(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ (0 i origo) (fra eks (ii) forrige forelesn.)

$$\partial_x F(0,0) = \lim_{x\to 0} \frac{F(x,0)-F(0,0)}{x} = 0 = \lim_{y\to 0} \frac{F(0,y)-F(0,0)}{y} = \partial_y F(0,0).$$

$$\underset{\text{utenfor origo}}{\partial_x F(x,y)} = \frac{\big(y(x^2-y^2)+xy(2x)\big)(x^2+y^2)-xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$\partial_y \partial_x F(0,0) = \lim_{y\to 0} \frac{\partial_x F(0,y)-\partial_x F(0,0)}{y} = \left[\begin{array}{l}\text{del uttrykket ovenfor med } y, \\ \text{la } x=0.\end{array}\right]$$

$$= \lim_{y\to 0} \frac{-y^4}{y^4} = -1.$$

Obs! Dette er ikke $\partial_y \partial_x F$ utenfor origo, kun i det punktet.

$$\underset{\text{utenfor origo}}{\partial_y F(x,y)} = \frac{\big(x(x^2-y^2)+xy(-2y)\big)(x^2+y^2)-xy(x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$\partial_x \partial_y F(0,0) = \lim_{x\to 0} \frac{\partial_y F(x,0)-\partial_x F(0,0)}{x} = \lim_{x\to 0} \frac{x^4}{x^4} = 1.$$

Nå: Funksjoner $U \subset \mathbb{R}^n \to \mathbb{R}^m$.

La oss begynne med $\mathbb{R}^2 \to \mathbb{R}^2$.

Eks. $F : [0,\infty) \times [0,2\pi) \longrightarrow \mathbb{R}^2$

$$(r,\theta) \longmapsto F(r,\theta) = (\overbrace{x(r,\theta)}^{F_1}, \overbrace{y(r,\theta)}^{F_2})$$

$$= (r\cos(\theta), r\sin(\theta)).$$

En funksjon!

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

- Er den kont?

  Ja, fordi komponentene $F_1 : \mathbb{R}^2 \to \mathbb{R}$,
  $F_2 : \mathbb{R}^2 \to \mathbb{R}$,
  er begge kontinuerlige.

- Er den partielt deriverbar?

  Ja, fordi: $\dfrac{\partial F}{\partial r} = \left(\dfrac{\partial F_1}{\partial r}, \dfrac{\partial F_2}{\partial r}\right) = (\cos(\theta), \sin(\theta))$

  og $\dfrac{\partial F}{\partial \theta} = \left(\dfrac{\partial F_1}{\partial \theta}, \dfrac{\partial F_2}{\partial \theta}\right) = (-r\sin(\theta), r\cos(\theta))$

- Er F kont. deriverbar?

  Ja, fordi: $\dfrac{\partial F}{\partial r}$, $\dfrac{\partial F}{\partial \theta}$ er kont. i (r,θ).

- Hva er tilsvarigheten til DF?

Skriver $\boxed{DF}$ eller J : $\left[\left[\frac{\partial F}{\partial r}\right]\left[\frac{\partial F}{\partial \theta}\right]\right] = \begin{bmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \theta} \end{bmatrix}$

'Jacobian'

en matrise!

Generelt: for $F: \mathbb{R}^n \to \mathbb{R}^m$, er

$$DF = m \underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}}_{n}$$

I vårt tilfelle:

$$DF(r,\theta) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

$\det DF(r,\theta) = r\cos^2(\theta) - (-r\sin^2(\theta)) = r \neq 0$ utenom i origo. Husk dette, kobler til $(r,\theta) \mapsto (x,y)$ ikke inverterbar i origo.

Hva er bra med Jacobimatrisen?

Kjerneregel $F: \mathbb{R}^n \to \mathbb{R}^m$, $g: \underline{\mathbb{R}} \to \mathbb{R}^n$

$F \circ g: \mathbb{R} \to \mathbb{R}^m$. Tenk på F som m kopier

$F_j: \mathbb{R}^n \to \mathbb{R}$, $F = (F_1, F_2, \dots, F_m)$.

$$\underset{\mathbb{R} \to \mathbb{R}}{(F_j \circ g)'} = \underset{\nabla F_j \cdot \gamma'}{\nabla F_j(g) \cdot g'} = [\nabla F_j(g)]\begin{bmatrix} g' \end{bmatrix}$$

$$(F \circ g)' = \begin{bmatrix} DF(g) \end{bmatrix}\begin{bmatrix} g' \end{bmatrix} = \underbrace{m\underbrace{\begin{bmatrix} \nabla F_1(g) \\ \vdots \\ \nabla F_m(g) \end{bmatrix}}_{n}\underbrace{\begin{bmatrix} g_1' \\ \vdots \\ g_n' \end{bmatrix}}_{1} n}_{\boxed{m \times 1}}$$

Hvis nå $\underset{\mathbb{R}^r \to \mathbb{R}^n}{G} = (g_1(t_1, \dots, t_r), \dots, g_n(t_1, \dots, t_r))$

blir $\dfrac{\partial (F \circ G)}{\partial t_j} = \begin{bmatrix} DF(G) \end{bmatrix}\begin{bmatrix} \dfrac{\partial G}{\partial t_j} \end{bmatrix}$

Helt likt
$t = t_j$!
ovenfor

og hele $D(F\circ G) =_{m} \underset{n}{[DF(G)]}\,\underset{r}{[DG]}_{n}$ $\boxed{m \times r}$

blir r kopier.

Tilbake til vårt eksempel:

Dersom vi har en funksjon $F(x,y)$ og ønsker bytte til polarkoord.,

$$\boxed{\tilde{F}(r,\theta) = F(x(r,\theta), y(r,\theta))},$$

Hva er $D\tilde{F}(r,\theta)$ dersom vi kjenner $DF(x,y)$?

La $G(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos(\theta), r\sin(\theta))$

$\mathbb{R}^2 \to \mathbb{R}^2$

$$\Rightarrow \boxed{\tilde{F} = F\circ G}$$

$$\nabla\tilde{F} = D(F\circ G) = [\nabla F\circ G]\underbrace{[DG]}_{\begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}}$$

F. eks $F(x,y) = x e^{y}, \quad \mathbb{R}^2 \to \mathbb{R}$

$$\nabla F(x,y) = (e^{y}, x e^{y}) = e^{y}(1, x)$$

$$(\nabla F)\circ G(r,\theta) = \underline{(1, r\cos(\theta))\, e^{r\sin(\theta)}}$$

$$\nabla(F\circ G)(r,\theta) = \left[\underbrace{(\nabla F)\circ G(r,\theta)}_{2}\right] \underset{2}{\left[\; DG \;\right]}_{2}$$

$$= e^{r\sin(\theta)} \begin{bmatrix} \cos(\theta) + r\cos(\theta)\sin(\theta) \\ -r\sin(\theta) + r^2\cos^2(\theta) \end{bmatrix}^{t} .$$

Sjekk: $r\cos(\theta)\, e^{r\sin(\theta)} = \tilde{F}(r,\theta)$

$\partial_r \rightsquigarrow (\cos(\theta) + r\cos(\theta)\sin(\theta))\, e^{r\sin(\theta)}$

$\partial_\theta \to (-r\sin(\theta) + r^2\cos^2\theta)\, e^{r\sin(\theta)}$

12.8 Implisitt derivasjon

Derivasjon som om en variabel varierte med en annen.

Eks. $\boxed{y - x^2 = 0.}$ Definerer dette en funksjon $y(x)$?

Prøv: Lat som om $y = y(x)$ og derivér:

$$\frac{d}{dx}: \quad y'(x) - 2x = 0 \iff y'(x) = 2x.$$

Definerer $\boxed{\quad}$ en funksjon $x = x(y)$?

Prøv: Lat som om $x = x(y)$.

$$\frac{d}{dy}: \quad 1 - \overset{\text{kj.reg.}}{2x(y)x'(y)} = 0$$

$$\iff \boxed{x'(y) = \frac{1}{2x(y)}} \qquad \boxed{\text{veldef. for } x \neq 0}$$

obs! at løsn. til $x' = \frac{1}{2x}$ er $x = \pm\sqrt{y} + C$

presis som forventet.

Finner to 'grener': $y - x^2 = 0$ kan løses

som $x = x(y) = \pm\sqrt{y}$ kun for $x \geq 0$

eller $x \leq 0$.

Senere: Implisitt derivasjon alltid OK

dersom den deriverte $\neq 0$.

Formel i to variabler: $F(x,y(x)) = C$.

$$\Rightarrow \frac{d}{dx} F(x,y(x)) = \nabla F(x,y(x)) \cdot (1, y'(x))$$

kj.r.

$$\swarrow \frac{d}{dx} C$$

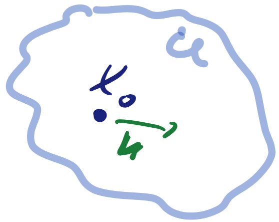
$$= 1 \cdot \frac{\partial F}{\partial x} + y' \cdot \partial_y F = 0$$

$$\Rightarrow \boxed{y'(x_0) = -\frac{\partial_x F(x_0,y_0)}{\partial_y F(x_0,y_0)}}$$

i et punkt (x_0,y_0) med $F(x_0,y_0) = C$.

12.9. Taylors formel / Taylorrekker

$F: \overset{\text{åpen}}{U} \subset \mathbb{R}^n \to \mathbb{R}$ der. bar, $x_0 \in U$



vektorer

$$F(x_0 + h) = \underbrace{F(x_0) + \nabla F(x_0) \cdot h}_{\text{linearisering}} + |h|\underline{\varepsilon(h)}$$

$\to 0$ da $h \to 0$

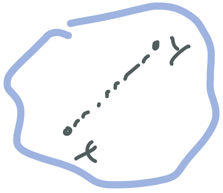
kont. i x_0 dersom $F \in C^1$

Dette er en Taylorformel.

'Minste Taylor' (Middelverdisetn.)

F deriverbar $\bar{U} \subset \mathbb{R}^n \to \mathbb{R}$, $x, y \in \bar{U}$,

med $\gamma(t) = (1-t)x + ty \subset \bar{U}$, $t \in [0,1]$.



Da $\exists\, \tau \in (0,1)$ og $c = \gamma(\tau)$:

$$\boxed{F(y) - F(x) = \nabla F(c) \cdot (y-x)}$$

vektorer!

Merk: Med $x = x_0$, $y = x_0 + h$, får vi:

$$\boxed{F(x_0+h) = F(x_0) + \nabla F(c) \cdot h}$$

Bevis: La $f := F \circ \gamma : [0,1] \to \mathbb{R}$

deriverbar med $f'(t) = \nabla F(\gamma(t)) \cdot \gamma'(t)$

$$= \nabla F(\gamma(t)) \cdot (y-x).$$

$$\boxed{\gamma(t) = (1-t)x + ty}$$

Middelv. setn. $\mathbb{R} \to \mathbb{R}$:

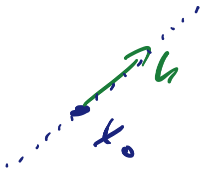
$$\exists\, \tau \in (0,1): \underbrace{f(1)}_{F(y)} - \underbrace{f(0)}_{F(x)} = f'(\tau)(1-0) = \nabla F(\overset{c}{\gamma(\tau)}) \cdot (y-x).$$

$$\Longrightarrow \quad F(y) - F(x) = DF(c)\cdot(y-x).$$

---

obs! Finnes ikke for $F:\mathbb{R}^n \to \mathbb{R}^m$. Kan ikke finne en felles τ for alle F_j.

Nå: la $\gamma(t) = \vec{x}_0 + t\vec{h}$ og bruk samme idé som ovenfor.

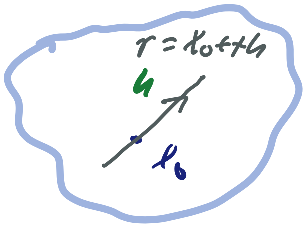


$\rightsquigarrow$ Taylor for $F: U \subset \mathbb{R}^n \to \mathbb{R}$.

Def. $F \in C^k(U,\mathbb{R}) \overset{\text{def.}}{\Longleftrightarrow} \underbrace{\partial x_i \dots \partial x_j}_{k \text{ stykk}} F$ kont.

Eks. $F \in C^2(\mathbb{R}^2,\mathbb{R})$ dersom $F''_{xx}, \underline{F''_{xy}, F''_{yx}}, F''_{yy}$ er kont.

Kj.r. $\begin{cases} F \in C^k \\ \gamma \in C^\infty \end{cases} \Longrightarrow f = F \circ \gamma \quad C^k(\mathbb{R},\mathbb{R})$



Taylor $\mathbb{R} \to \mathbb{R}$:

$$f(t) = f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \dots + \frac{t^k}{k!} f^{(k)}(\tau)$$

τ mellom 0 og t.

$$\text{Men } f(0) = F(r(0)) = F(x_0)$$

$$f'(0) = F(r)'(0) = \nabla F(r(0)) \cdot \dot{r}(0)$$

$$= \nabla F(x_0) \cdot \underline{h}$$

Så for $t = 1$ blir begynnelsen på Taylorrekken:

$$\boxed{\underbrace{F(x_0+h)}_{f(1)} = \underbrace{F(x_0) + h \cdot \nabla F(x_0)}_{\text{Taylorpolynom av grad 1}} + \cdots}$$

Kan fortsette så langt F er kont. deriverbar.

Men hva er $f''(0)$?

Se på $F(x_1, x_2)$, $h = (h_1, h_2)$, $r(t) = (x_1 + th_1, x_2 + th_2)$

$$f' = \nabla F(r) \cdot \dot{r} = \left[(\partial_{x_1} F)(r), (\partial_{x_2} F)(r)\right] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= \boxed{h_1 \partial_{x_1} F(r) + h_2 \partial_{x_2} F(r).}$$

$$f'' = h_1 \nabla(\partial_{x_1} F)(r) \cdot \dot{r} + h_2 \nabla(\partial_{x_2} F)(r) \cdot \dot{r}$$

$$= h_1^2 \partial_{x_1}^2 F(r) + \underbrace{h_2 h_1 \partial_{x_2} \partial_{x_1} F(r) + h_1 h_2 \partial_{x_1} \partial_{x_2} F(r)}_{2 h_1 h_2 \partial_{x_1} \partial_{x_2} F(r)}$$

$$+ h_n^2 \partial_{x_n}^2 F(a)$$

Med $D^2 F = \begin{bmatrix} \partial_{x_1}^2 F & \partial_{x_1}\partial_{x_n} F \\ \partial_{x_n}\partial_{x_1} F & \partial_{x_n}^2 F \end{bmatrix}$ får

'Hessian'

$$F(x_0+h) = F(x_0) + h\cdot DF(x_0) + [h] \begin{bmatrix} D^2F \end{bmatrix} \begin{bmatrix} h \end{bmatrix} + O(|h|^3)$$

dersom $F \in C^3$

der $|O(|h|^3)| \le C|h|^3$

obs! <u>for små h</u>

C bestemmes av 3:e ordens deriverte til F nært x_0.

Taylorpolynomer er unike (har vi funnet et, er det det riktige.

Eks. Taylor av $\cos(x)\sin(y)$ i $(0,0)$.

Vet $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \frac{y^7}{7!} + \dots$

$$\Rightarrow \cos(x)\sin(y) = \underbrace{y}_{O(|(x,y)|)} - \underbrace{\frac{x^2y}{2} - \frac{y^3}{3!}}_{O(|(x,y)|^3)} + O(|(x,y)|^5)$$

Obs! Polynomer er sine egne Taylorutv. krings origo.

$$D^2F(x,y) = \begin{bmatrix} 0 & 2y \\ 2y & 2x \end{bmatrix}$$

Eks. $F(x,y) = x + xy^2$, $DF(x,y) = (1+y^2, 2xy)$

$$F(0+h_1, 0+h_2) = F(h_1,h_2) = \underline{h_1 + h_1 h_2^2}$$

$\leftarrow$ Fra D^3F...

$\uparrow$ $DF(0,0)\cdot(h_1,h_2)$

Men $F(1+h_1, 1+h_2) =$

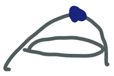
$$F(1,1) + (h_1,h_2)\cdot(2,2)\cdot O(|(h_1,h_2)|^2)$$

$$= \underbrace{2 + 2h_1 + 2h_2}_{} + O(h_1^2 + h_2^2)$$

TP til F gjennom $(1,1,2)$

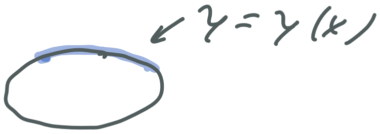
Grunnleggende egenskaper til kont. funksj.

- Ekstremalverdisetn.



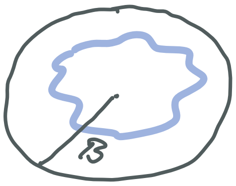
- Omvendte funksjonsteorem.

- Implisitte —"— —



Def. $U \subset \mathbb{R}^n$ begrenset $\overset{\text{def.}}{\iff}$

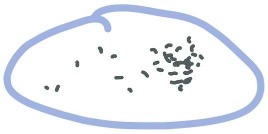
$\exists B > 0$; $|x| \le B$ $\forall x \in U$



Bolzano – Weierstrass

$U \subset \mathbb{R}^n$ begrenset $\implies$ hver følge $(x_j)_j \subset U$

har en konvergent delfølge.



Bevis B-W sann i én dimensjon. (MA1102)

$$\text{Skriv } \overline{x}_j = (\underset{}{\overset{\mathbb{R}\,\swarrow}{x_j^1}}, \overset{\searrow}{x_j^2}, \ldots, x_j^n) \in \mathbb{R}^n.$$

$(x_j^1)_j$ begr. følge i $\mathbb{R}$ $\implies$ har konv. delfølge i $\mathbb{R}$.

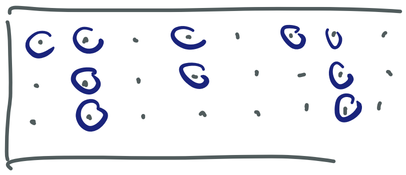
Betrakt x_j^2 for samme delfølge

$\xRightarrow{\text{B-W 1 dim}}$ har en konv. delfølge.

$\xRightarrow{\text{o.s.v.}}$ $\exists$ delfølge til $(\underset{\downarrow}{x_j^1}, \underset{\downarrow}{x_j^2}, \ldots, \underset{\downarrow}{x_j^n})$

$(x^1, x^2, \dots, x^n) = \vec{x}$

konv. i hver komponent.



$\Longrightarrow \exists\, x \in \bar{U} : x_i \to x.$

Heine-Borel U lukket og begrenset i $\mathbb{R}^n$

$\Longleftrightarrow$ U er kompakt,

dvs. hver følge $(x_i)_i \subset \bar{U}$ har en konv. delfølge : $x_{i_k} \xrightarrow{k\to\infty} x \in \bar{U}$.

Ekstremalverdisetningen.

$F : \bar{U} \subset \mathbb{R}^n \to \mathbb{R}$ med U kompakt (lukket og begrenset)

F kont på $\bar{U}$ $\Longrightarrow \exists\, x_0 \in \bar{U} : F(x_0) = \max\limits_{x\in\bar{U}} F(x)$.

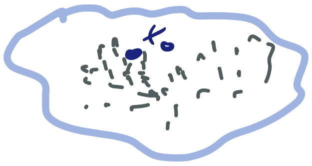
Bevis $\exists \sup\limits_{x\in\bar{U}} F(x)$, velg $(x_j)_j$ sånn at

$$\boxed{\lim_{j\to\infty} F(x_j) = \sup_{x\in\bar{U}} F(x)}$$

$(x_j)_j \subset \bar{U}$ kompakt $\xrightarrow[HB]{BW}$ $\exists$ konv

delfølge $(x_{j_k})_k$; $x_{j_k} \to x_0 \in \overline{\bar{U}} = \bar{U}$

U lukket.



F kont. $\boxed{F(x_{j_k}) \to \underset{\mathbb{R}}{F(x_0)}}$

da $x_{j_k} \to x_0$.

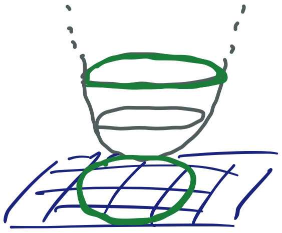
$\forall x \in \bar{U}$ gjelder $F(x) \leq \sup_{x \in \bar{U}} F(x) = \lim_{j\to\infty} F(x_j)$

$$= \lim_{k\to\infty} F(x_{j_k}) = F(x_0).$$

Så $F(x_0) = \max_{x \in \bar{U}} F(x)$.

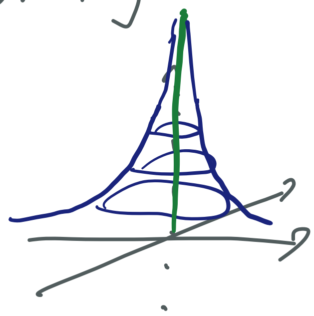
#

Tenk på : $F(x,y) = x^2 + y^2$



kompakt viktig

$\frac{1}{|(x,y)|}$



Nå: IFT (Inverse/Implicit Function Theorem)

Disse er ekvivalente.

Def. $\boxed{\mathrm{id}: x \mapsto x.\ \mathbb{R}^n \to \mathbb{R}^n}$

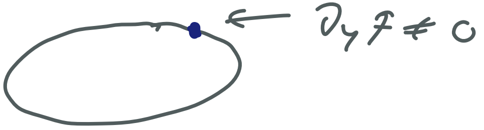
Inverse f.sjon, idé: $f:\mathbb{R}\to\mathbb{R},\ f'(x_0)\neq 0$

$\implies \exists f^{-1}$ nært $y_0 = f(x_0)$, og

$\frac{d}{dx}$ på $f^{-1}\circ f = \mathrm{id}$

$$\implies \left((f^{-1})'\circ f\right)\cdot f' = 1 \implies \boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}$$

Implisitte, idé: $\boxed{F(x,y)=0,\ \partial_y F(x_0,y_0)\neq 0}$



$F:\mathbb{R}^2\to\mathbb{R}$

$\partial_y F$ kont $\implies \partial_y F(x,y)\neq 0$ nært (x_0,y_0).

Betrakt x som en parameter.

Omvendte f.sjon. i én variabel

$\Longrightarrow$ $\forall x$ nært x_0 $\exists\, \Phi_x : F \longrightarrow y,$

dvs $y = \Phi_x(F)$ der $F(x,y) = 0 \Longrightarrow \boxed{y = \Phi_x(0)}$

Så y bestemt av x nært x_0.

$F(x,y) = 0 \iff \underline{F(x, y(x)) = 0}$ nært (x_0, y_0)

$$\frac{d}{dx} \overset{\text{kj.r.}}{\leadsto} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\, y'(x) = 0$$

$$\Longrightarrow y'(x) = -\frac{\partial F/\partial x}{\partial F/\partial y}$$ i (x,y) på kurven.

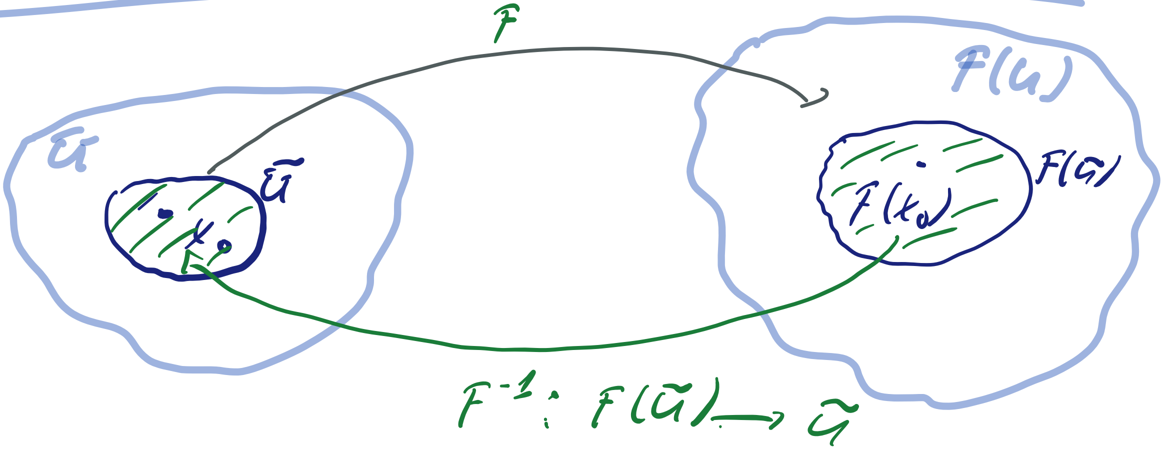
Inverse funksjonssetningen for $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$

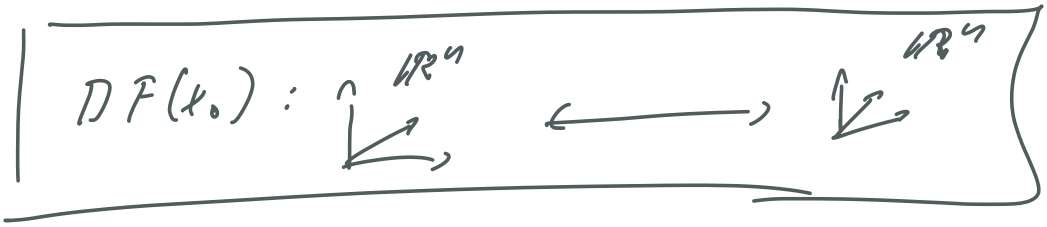
$U \subset \mathbb{R}^n$ åpen, $x_0 \in \bar{U}$ og $F \in C^1(\bar{U}, \mathbb{R}^n)$

med $\underbrace{DF[x_0]}_{n\times n\text{-Jakobi matrise}}$ inverterbar $\mathbb{R}^n \to \mathbb{R}^n$.

$\Longrightarrow$ $\exists\, \tilde{U} \ni x_0 : F : \tilde{U} \longrightarrow F(\tilde{U})$ er inverterbar, og

$$\boxed{D(F^{-1}) = (DF)^{-1} \circ F^{-1} \text{ kont på } F(\tilde{U}).}$$





Implisitte f. teor. for $F: U \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$

$U \subset \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ åpen, $u_0 = (\underset{\mathbb{R}^n}{\underset{\cap}{x_0}}, \underset{\mathbb{R}^m}{\underset{\cap}{y_0}}) \in U$,

$F \in C^1(U, \mathbb{R}^m)$ med $F(x_0, y_0) = \overset{\mathbb{R}^m}{0}$ og

$D_y F(x_0, y_0)$ omvendbar $\mathbb{R}^m \to \mathbb{R}^m$.

$\Longrightarrow \exists\ \tilde{U} = \tilde{B}_{x_0} \times \tilde{B}_{y_0} \subset \mathbb{R}^n \times \mathbb{R}^m$ og

$\Phi \in C^1(\tilde{B}_{x_0}, \tilde{B}_{y_0}) : \quad F(x, \Phi(x)) = 0$

inneholder alle løsninger til $F(x,y)=0$ i $\bar{U}_r$

og $D\Phi = -[D_y F(\Phi)]^{-1} D_x F(\Phi)$.

Bevis (gitt omvendte f.setn.)

$$\text{La } G : \underset{\mathbb{R}^n \times \mathbb{R}^m}{(x,y)} \longrightarrow \underset{\mathbb{R}^n \times \mathbb{R}^m}{(\overset{G_1}{x}, \overset{G_2}{F(x,y)})}$$

$$\Longrightarrow G \in C^1(\bar{U}, \mathbb{R}^{n+m}) \text{ med } \begin{bmatrix} 1 & \cdots & 0 \\ 0 & & 1 \end{bmatrix}$$

$$DG = \begin{bmatrix} D_x G_1 & D_y G_1 \\ D_x G_2 & D_y G_2 \end{bmatrix} = \begin{bmatrix} [Id] & [0] \\ [D_x F] & [D_y F] \end{bmatrix}$$

$$\Longrightarrow \det(DG) = \det(D_y F) \underset{\underline{\text{antatt inv.bar}}}{\neq} 0 \quad \text{i } (x_0, y_0).$$

omv. setn.

$$\Longrightarrow \quad \exists\ C^1\text{-invers}$$

$$G^{-1} : \underset{(x,\, F(x,y))}{(v,w)} \longrightarrow \left(\underset{x}{G_1^{-1}(v,w)},\ \underset{\boxed{y}}{G_2^{-1}(v,w)} \right)$$

$$y = G_2^{-1}(x, F(x,y)) = G_2^{-1}(x, 0) \text{ der } F(x,y)=0.$$

Så $y = \Phi(x) \overset{\text{def.}}{=} G_2^{-1}(x,0)$ nær (x_0, y_0).

Videre: $F(x, \Phi(x)) = 0$

$\overset{\text{kj.r.}}{\Longrightarrow} [D_x F] + [D_y F][D\Phi] = 0$

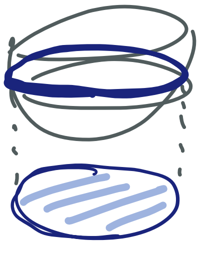
$\Longrightarrow [D\Phi] = -[D_y F]^{-1}[D_x F]$ i $(x, \Phi(x))$.

13.1 – 13.2 Lokale og globale ekstrema

$F: U \subset \mathbb{R}^n \to \mathbb{R}$ kont. på kompakt $K \subset \bar{U}$

$\Longrightarrow \exists\, x_0 \in K : F(x_0) = \max_{x \in K} F(x)$.

Hvordan finne x_0 / maks?

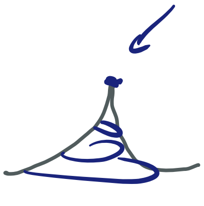


(i) x_0 kan tilhøre randen til K

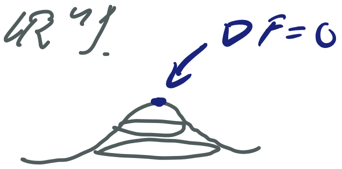
- Må sjekke $F(x)$; $x \in \partial K$.

(ii) x_0 kan være et punkt der F ikke er deriverbar.

- Må sjekke singulære punkter.



(iii) Ellers må $DF(x_0) = 0$ (i $\mathbb{R}^n$).



Bevis (for at (iii) er eneste andre mulighet)

x_0 indre punkt i $\bar{U}$, F deriverbar i x_0

$$\Longrightarrow \underbrace{F(x_0+h) - F(x_0)}_{(*)} = DF(x_0)\cdot h + |h|\,\underbrace{\varepsilon(h)}_{\substack{\to 0 \\ \text{då } h\to 0.}}$$

$$F(x_0) \text{ lok. maks} \overset{\text{def.}}{\Longleftrightarrow} \exists\,\delta;\ (*) \leq 0 \quad \forall\, |h| < \delta.$$

Si $\boxed{\frac{\partial F}{\partial x_j}(x_0) \neq 0}$. Velg $h = (0,\dots,0,\underset{\uparrow}{h_j},0,\dots,0)$;

$$|\varepsilon(h)| \leq \frac{1}{2}\left|\frac{\partial F}{\partial x_j}(x_0)\right|$$

$$\Longrightarrow \underbrace{\frac{F(x_0+h) - F(x_0)}{h_j}}_{\substack{\text{Veksler tegn med } h_j \\ \text{(eller helt lik 0)}}} = \overbrace{\frac{\partial F}{\partial x_j}(x_0) \pm \underbrace{\varepsilon(h)}_{\leq \frac{1}{2}\left|\frac{\partial F}{\partial x_j}(x_0)\right|}}^{\text{fikser fortegn, og } \neq 0}$$

Motsigelse $\Longrightarrow \frac{\partial F}{\partial x_j}(x_0) = 0 \quad \forall\, j = 1,\dots,n$

$$\Longrightarrow DF(x_0) = 0 \qquad ↯$$

Obs! $\underline{\nabla F = 0}$ nødvendig for lok. maks/min, men ikke tilstrekkelig.

<u>kritisk punkt</u>

<u>stasjonært punkt</u>

Si $F \in C^2(U, \mathbb{R})$, $\nabla F(x_0) = 0$.

$$\Rightarrow \underline{F(x_0+h) - F(x_0)} = \frac{1}{2} D^2 F(c)(h,h)$$

$$\boxed{\quad\quad\quad} = [\,h\,] \begin{bmatrix} D^2 F(c) \end{bmatrix} \begin{bmatrix} h \end{bmatrix}$$ kvadratisk form.

Hessematrise

<u>Positivt/negativt definit</u> dersom alle <u>egenverdier til $D^2F[c]$ er strengt $\overset{>0}{\text{positive}}$,</u> <u>eller alle strengt $\overset{<0}{\text{negative}}$.</u> $[h_1 \;\; h_2]\begin{bmatrix} F_{x_1 x_1} & F_{x_1 x_2} \\ F_{x_2 x_1} & F_{x_2 x_2} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

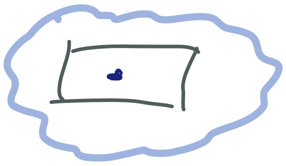
<u>I $\mathbb{R}^2$:</u> $$\boxed{\quad\quad} = \boxed{h_1^2 \partial_{x_1}^2 F + 2 h_1 h_2 \partial_{x_1} \partial_{x_2} F + h_2^2 \partial_{x_2}^2 F.}$$

$\geq \lambda_{min} |h|^2$

$\leq \lambda_{max} |h|^2$

D^2F kont, $c \xrightarrow{h \to 0} x_0$, <u>Egenverdiene til $D^2F(x_0)$</u> <u>avgjør.</u>

Teorem (Andrederivattesten)



$F \in C^2(U, \mathbb{R})$, $K \subset U \subset \mathbb{R}^2$, $x_0 \in K$.

åpen (U) kompakt (K)

$\nabla F(x_0) = 0$, $F_{xx} F_{yy} - (F_{xy})^2 > 0$ (< 0) i x_0

$\Longrightarrow$ F har lokalt ekstremum i x_0

(F har saddelpunkt i x_0)

Obs! ≥ 0 (≤ 0) gir ingen utsagn.

Bevis

$$D^2 F = \begin{bmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{bmatrix}$$

$$\Longrightarrow \det(D^2 F) = F_{xx} F_{yy} - (F_{xy})^2 = (*)$$

Men vet også: $\det D^2 F[x_0] = \lambda_1 \lambda_2$

egenverdier

likt fortegn

$\lambda_1 \lambda_2 > 0 \Longleftrightarrow (*) > 0$

$\lambda_1 \lambda_2 < 0 \Longleftrightarrow (*) < 0.$

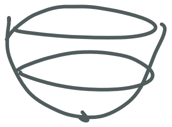
ulike fortegn

∎

Huskeregel

$$\boxed{F_{xx}F_{yy} - F_{xy}^2}$$

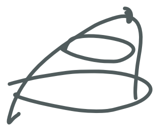
lok. min



- x^2+y^2 $\quad D^2F(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$DF(0,0) = (0,0)$

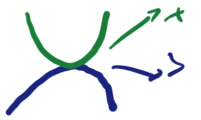
lok. maks.



- $-x^2-y^2$, $\quad D^2F(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

- x^2-y^2, $\quad D^2F(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

sadelpunkt



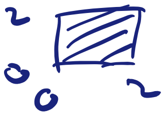
Eks. oppg. 1 $\quad$ 20/8 $\quad$ 2020

$$f(x,y) = 4xy - 2x^2 - y^4$$

(i) Finn alle kritiske punkter til f i $\mathbb{R}^2$.

(ii) Hvilke av disse er lok. maks/min/sadelpunkter?

(iii) Hva er $\max_K f$, $K = \{(x,y) : 0 \le x \le 2, 0 \le y \le 2\}$



Løsn. (i) kritisk punkt $\Longleftrightarrow$

$$Df(x,y) = (4y - 4x,\ 4x - 4y^3) = (0,0)$$

$$\iff \begin{cases} x = y \\ x = y^3 \end{cases} \iff \begin{cases} x = y \\ x(1-x^2) = 0 \end{cases}$$

$$\iff \begin{cases} (x,y) = (0,0) \ \text{eller} \\ (x,y) = \pm(1,1) \end{cases}$$

Svar: kritiske punkter er (0,0), (1,1) og (-1,-1).

(ii) $$D^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 4 & -12y^2 \end{bmatrix}$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{(0,0)} = (-4)(0) - 4\cdot 4 = -16 < 0$$

$$f_{xx}f_{yy} - f_{xy}^2 \Big|_{\pm(1,1)} = (-4)(-12) - 4\cdot 4 = 32 > 0.$$

Andrederivertesten ⇝ (0,0) sadelpunkt
$\pm(1,1)$ lok. ekstrema.

$f_{xx} = -4 < 0 \implies$ lok. maks i $\pm(1,1)$.

(iii) Polynomer kont + K kompakt (lukket og begr.)

$$\xRightarrow{\text{ekstremalv. setn.}} \quad \exists \max_K f$$

$$f \text{ kont. deriverbar} \implies \boxed{\text{maks på } \partial K \text{, eller der } Df = (0,0)}$$

på randen

~~$f(0,0) = 0$~~, $f(1,1) = 1$ eneste mulighet i det indre av K

∂K: • $f|_{x=0} = -y^4$ maks i $y=0$, $f(0,0)=0$.

• $f|_{x=2} = 8y - 8 - y^4$ finn maks på $y \in [0,2]$

$f(2,0) = -8$, $f(2,2) = -8$.

mulig maks på $0<y<2$ der $x=2$:

$$\frac{d}{dy} f(2,y) = 8 - 4y^3 > 0 \text{ på } 0<y<2$$

$$\Rightarrow \text{maks } f|_{x=2} = f(2,2) = -8 < 0.$$

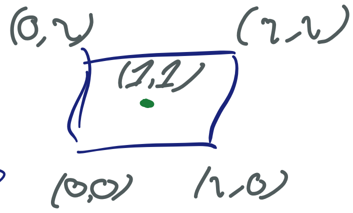
• $f|_{y=0} = -2x^2 \leq 0$

• $f|_{y=2} = 8x - 2x^2 - 16$ finn maks på $x \in [0,2]$

$f(0,2) = -16$. $f(2,2) = -8$.

$$\frac{d}{dx} f(x,2) = 8 - 4x > 0 \text{ på } 0<x<2$$

$$\Rightarrow \text{maks}_{0\leq x\leq 2} f(x,2) = -8 < 0.$$



$\Rightarrow$ maks $f \leq 0$ på ∂K.

$$\boxed{\text{Så maks}_K\, f = 1 \text{ i } (1,1).}$$

Kortere: kvadratkomplettere f!

$$4xy - 2x^2 - y^4 = -2\underbrace{(x^2-2xy)}_{(x-y)^2-y^2} - y^4$$

$$= -2(x-y)^2 - \underbrace{(y^4-2y^2)}_{(y^2-1)^2-1}$$

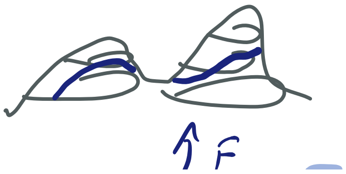
$$= 1 - 2(x-y)^2 - (y^2-1)^2 \le 1$$

med likhet eksakt ; $x = y = \pm 1$.

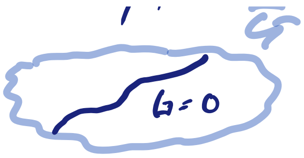
globalt maks på $\mathbb{R}^2$; $(x,y) = \pm(1,1)$.

13.3 Minimering ved bivilkår / Lagrange multiplikatorer

La $F : U \subset \mathbb{R}^n \to \mathbb{R}$



og $G : U \subset \mathbb{R}^n \to \mathbb{R}$.



Hva gjelder for min/maks av F over $\{x \in \bar{U} : G(x) = 0\}$?

Spørsmål 1: Har $\{G(x) = 0\}$ struktur?

IFT: $|\nabla G| \neq 0$ langs $G(x) = 0$ $\Longrightarrow$

$\exists$ C^1-funksjon slik at $\{G(x)=0\}$ er en kurve / flate / $(n-1)$-dimensjonal hyperflate.

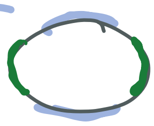
Eks $\mathbb{R}^2$: $G(x_1, x_2)$ med $\frac{\partial G}{\partial x_1} \neq 0$

lokalt $\Longrightarrow$ $G(x_1, x_2) = 0 \Longleftrightarrow G(\phi(x_2), x_2) = 0$ kurve!

Eks $\underbrace{x_1^2 + x_2^2 - 1}_{G(x_1, x_2)} = 0$. $|\nabla G| = |(2x_1, 2x_2)|$

$= \sqrt{4x_1^2 + 4x_2^2} = 2 \neq 0$

$\Longrightarrow$ Enten $x_1 = x_1(x_2)$ eller $x_2 = x_2(x_1)$



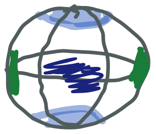
Eks. $\mathbb{R}^3$ $G(x_1, x_2, x_3)$ med $\frac{\partial G}{\partial x_1} \neq 0$

flate!

lokalt $\Longrightarrow$ $G(x_1,x_2,x_3)=0 \iff G(\phi(x_2,x_3),x_2,x_3)=0$

Eks: $\underbrace{x_1^2+x_2^2+x_3^2-1}_{G}=0$, $|\nabla G| = |(2x_1,2x_2,2x_3)|$
$= 2 \neq 0$

$\Longrightarrow$ $x_1 = x_1(x_2,x_3)$, $x_2 = x_2(x_1,x_3)$ eller
$x_3 = x_3(x_1,x_2)$.



La γ være en C^1-kurve på $\{G(x)=0\}$.

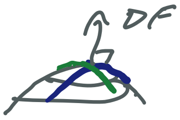
$\gamma: I \to U \subset \mathbb{R}^n$, La $\gamma(0)=x_0$.

Dersom F har et lokalt maks/min i x_0 over $\{G(x)=0\}$ må $\frac{d}{dt}(F\circ\gamma)(0)=0 \overset{\text{kj. reg.}}{\iff}$

$\nabla F(\overset{x_0}{\gamma(0)})\cdot\dot\gamma(0)=0 \iff \nabla F \perp \dot\gamma$ for

Tangentvektor!

hver γ over $\{G(x)=0\}$ gjennom x_0.



$\{G(x)=0\}$ $(n-1)$-dimensjonal

$\Longrightarrow$ ∇F normal til $\{G=0\}$.

Men vet også $G(\gamma(t))=0 \xrightarrow[\text{kj. reg.}]{d/dt} \nabla G(\gamma)\cdot\dot\gamma=0$

$\forall \gamma$ på $\{G=0\}$.

$\Longrightarrow$ ∇G normal til $\{G=0\}$.

Så $\boxed{\nabla F \,/\!/\, \nabla G \text{ i } x_0.}$, dvs.

$$\boxed{\exists \lambda \in \mathbb{R} : \quad \begin{array}{l} \nabla F(x_0) = \lambda \nabla G(x_0) \\ \text{eller } \nabla G(x_0) = 0 \end{array}}$$

Lagrangemultiplikator

Setning (Lagrangemult.) Et lok maks/min for $F \in C^1(U, \mathbb{R})$ over $\{x \in \bar{U} ; G(x) = 0\}$, $G \in C^1(U, \mathbb{R})$, $U \overset{\text{åpen}}{\subset} \mathbb{R}^n$, og $\nabla G \neq 0$ på må realiseres i et punkt der $F - \lambda G$ har et kritisk punkt: $\boxed{\nabla F = \lambda \nabla G}$ for noen $\lambda \in \mathbb{R}$.

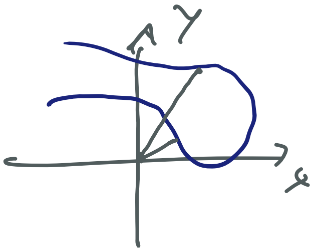
Merk: Må også sjekke punkter der $\nabla G = 0$ på $\{G(x) = 0\}$.

Eks. Oppgave 5, 8.6 2022

Finn største / minste avstand fra origo til kurven $5x^2+6xy+5y^2=8$.

Løsn. Minimér / maksimér $\boxed{F(x,y)=x^2+y^2}$

over $G(x,y)=5x^2+6xy+5y^2-8=0$.



$F, G \in C^2 \rightsquigarrow$ nødvendig vilkår:

$\nabla F \parallel \nabla G$, dvs.

$\nabla F = \lambda \nabla G$ eller $\nabla G = (0,0)$ i min/maks.

$\nabla F(x,y) = (2x, 2y) = \lambda(10x+6y, 10y+6x) = \lambda \nabla G$

$\lambda = 0 \Rightarrow (x,y) = (0,0)$ umulig (ligger utenfor $G(x,y)=0$)

$$\begin{cases} 2x = 10\lambda x + 6\lambda y & \cdot y \\ 2y = 10\lambda y + 6\lambda x & \cdot(-x) \end{cases}$$

Obs! $y=0 \Rightarrow x=0$
$x=0 \Rightarrow y=0$
ikke på kurven!

$0 = 0 + 6\lambda(y^2-x^2) \underset{\lambda\neq0}{\iff} x^2=y^2 \Leftarrow x=\pm y$.

Sett inn: $G(x,x) = 16x^2-8=0 \Leftarrow \boxed{x=\pm\frac{1}{\sqrt{2}}}$

$G(x,-x) = 4x^2-8=0 \iff \boxed{x=\pm\sqrt{2}}$

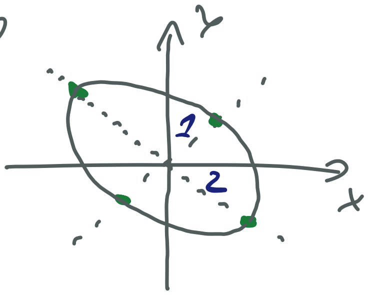
$F(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2} = 1$ lok. min Avstand: $\sqrt{1} = 1$

$F(\pm \sqrt{2}, \mp \sqrt{2}) = 2 + 2 = 4$ lok. maks Avstand: $\sqrt{4} = 2.$

Obs! $DG(x,y) = (10x + 6y, 10y + 6x) = (0,0)$

$$\iff \begin{cases} 5x = -3y \\ 5y = -3x \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$ ikke på $G(x,y)=0$

Alt. $$\begin{cases} x = \frac{1}{\sqrt{2}}(u+v) \\ y = \frac{1}{\sqrt{2}}(u-v) \end{cases} \iff \begin{cases} u = \frac{1}{\sqrt{2}}(x+y) \\ v = \frac{1}{\sqrt{2}}(x-y) \end{cases}$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{DF} \begin{pmatrix} v \\ u \end{pmatrix}$$

inv. har med $(DF)^{-1} = (DF)^t$

ON-transf. rotasjon $-\frac{\pi}{4}$ $\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$

$5x^2 + 6xy + 5y^2 = 8u^2 + 2v^2 = 8$

Så: $G(x,y) = 0 \iff \boxed{u^2 + \left(\frac{v}{2}\right)^2 = 1}$

ellipse med halvakser

1 i retning u,

2 i retning v.

Bevis omvendte f.teorem (M. Spivak, 1940–2020)

La $A = DF(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ med $\det(A) \neq 0$.
invertibel matrise etc.

$$D(\underbrace{A^{-1}F}_{G})(x_0) = A^{-1}DF(x_0) = A^{-1}A = Id$$

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

Dersom $\exists\, G^{-1}$; så $\exists\, F^{-1} = G^{-1} \circ A^{-1}$:

$$F \circ F^{-1} = A \underbrace{G \circ G^{-1}}_{id} \circ A^{-1} = A A^{-1} = id.$$

Så vi viser setningen for G ($\rightsquigarrow F$).

med $DG(x_0) = id = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$.

$$\boxed{D(F^{-1}) = D(G^{-1} \circ A^{-1})}$$

G C^1: $G(x_0+h) = G(x_0) + \underbrace{DG(x_0)\,h}_{\begin{bmatrix} 1 & & 0 \\ 0 & \ddots & 1 \end{bmatrix}\begin{bmatrix} h \end{bmatrix}} + |h|\underbrace{\varepsilon(h)}_{\to 0 \text{ da } h \to 0}$

$$\underset{\exists\, \delta > 0}{\Longrightarrow} \quad \frac{|G(x_0+h) - G(x_0)|}{|h|} \neq 0 \quad \text{når } 0 < |h| < \delta.$$

$\Longrightarrow$ $G(x) \neq G(x_0)$ for x nært x_0; $x \in B_\delta(x_0)$

$$B_\delta(x_0) = \{x : |x - x_0| < \delta\}.$$

$G \; C^1 \iff DG$ kont $\implies DG(x) = Id + \underbrace{A(x)}_{\to 0}$

da $x \to x_0$.

Nå: Viser at G er injektiv på $B_\delta(x_0)$.

Betrakt $|(G(x)-x)-(G(y)-y)|$

$$\overset{\Delta\text{-ulikh.}}{\leq} \sum_{j=1}^{n} |(G_j(x)-x_j)-(G_j(y)-y_j)|$$

$$= \sum_{j=1}^{n} \Big| \underbrace{\nabla(G_j(x)-x_j)}_{1+A_j(x)|_{x=c}-1}\Big|_{x=c} \Big| \, |x-y|$$

$$\leq \underbrace{\sum_{j=1}^{n} |A_j(c)|}_{\leq \frac{1}{2}} |x-y|$$

$\leq \frac{1}{2}$ dersom x, y nært x_0, dvs velg δ tilstr. liten

Men har også: $|G(x)-x-(G(y)-y)|$

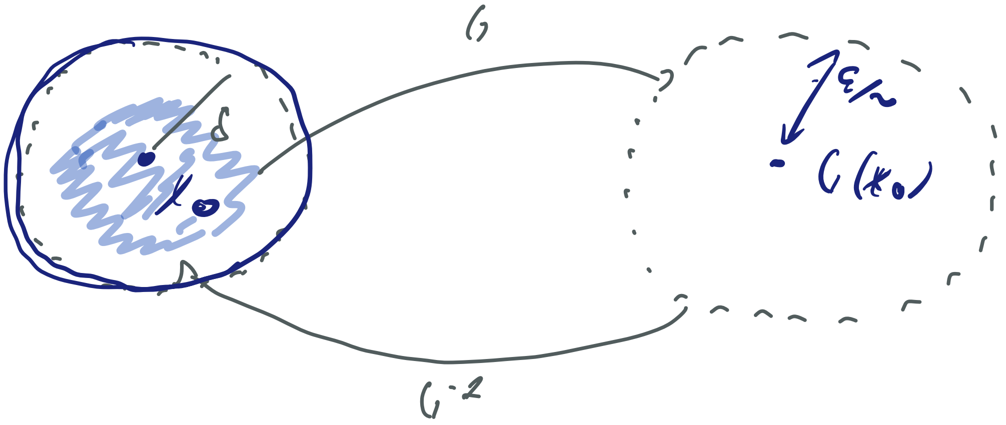
$$\overset{}{\geq} |x-y| - |G(x)-G(y)|$$

∇-ulikh.

$$\Longrightarrow \boxed{\frac{1}{2}|x-y| \leq |G(x)-G(y)|}$$

G injektiv på $B_\delta(x_0)$: $x \neq y \Longrightarrow G(x) \neq G(y)$.

- G surjektiv? (dvs på hva?)



$x \longmapsto |G(x) - G(x_0)|$ kont.

$\{|x - x_0| = \delta\}$ **kompakt**

Ekstr. verdi. setn. $\Longrightarrow$

$$\underbrace{\exists \min_{x \in \partial B_\delta(x_0)} |G(x)-G(x_0)|}_{\varepsilon} > 0$$

Betrakt $B_{\varepsilon/2}(G(x_0))$

Vil vise $\forall y \in B_{\varepsilon/2}(G(x_0))$ $\overset{\text{unik}}{\exists !}$ $x \in B_\delta(x_0)$;

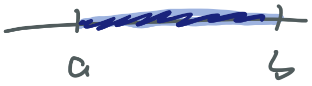
$G(x)=y$.

Hvordan: Fiksér y og betrakt avstanden $h(x) = |y - G(x)|^2$.

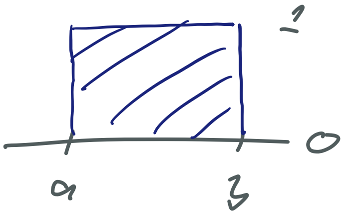
• • •

14.1 - 14.3 Integrasjon i $\mathbb{R}^2$ $(\mathbb{R}^n)$

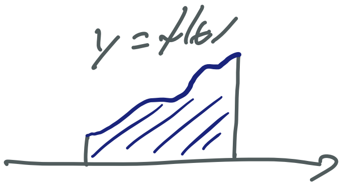
Idé: (i) $\int_a^b dx = b-a$ beskriver avstanden mellom a og b (gitt $a \leq b$).



(ii) Men også: $\int_a^b 1\,dx$ et areal til rektangelen



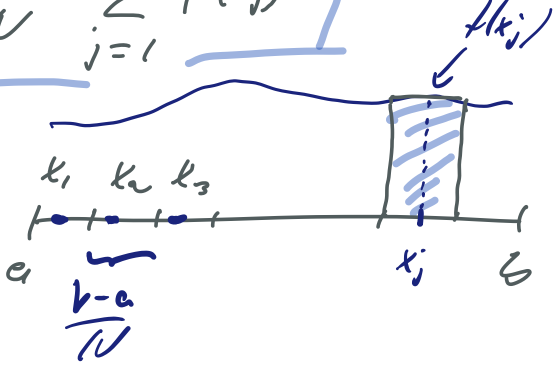
(iii) Generelt: $\int_a^b f(x)\,dx$ beskriver arealet mellom $y=f(x)$ og x-aksen.



Hvordan gjøres dette for $f: I \subset \mathbb{R} \to \mathbb{R}$?

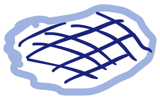
$$\boxed{\int_a^b f(x)\,dx = \lim_{N\to\infty} \frac{b-a}{N} \sum_{j=1}^{N} f(x_j)}$$

dersom f er integrerbar.



Hva svarer dette til for $f: U \subset \mathbb{R}^2 \to \mathbb{R}$?

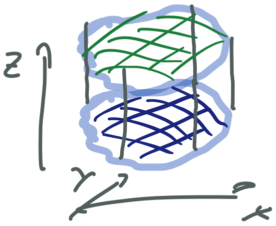
(Merk: U er 'fylt', ik utenfor.)



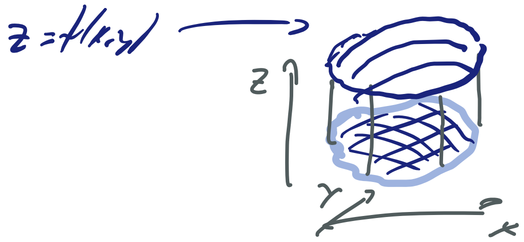
(i) $\boxed{\iint_{\bar U} dx\,dy}$ beskriver arealet til $\bar U$.

Kan også skrives $\iint dA$, $\iint dA(x,y)$, $\int dA$.

(ii) $\boxed{\iint_{\bar U} 1\,dx\,dy}$ volumet med base $\bar U$ og høyde 1.

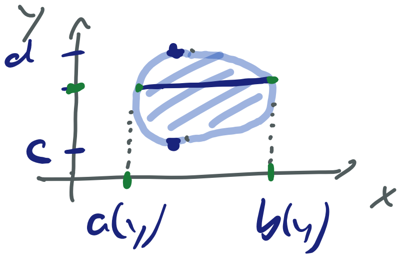


(iii) $\boxed{\iint_{\bar U} f(x,y)\,dx\,dy}$ volumet mellom flaten $z = f(x,y)$ og $\bar U$ i xy-planet.



Hvordan bestemme $\iint_{\Omega} dx\,dy$ matematisk?

Alt. 1

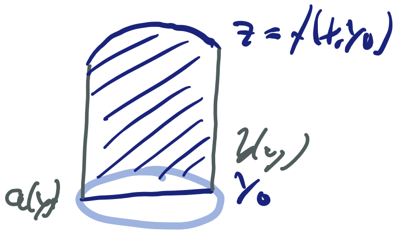


$\forall y \in [a,b]$ beregn avstanden $\int_{a(y)}^{b(y)} dx = b(y) - a(y).$

Deretter: integrér i y:

$$A(\Omega) = \int_c^d \left(\int_{a(y)}^{b(y)} dx \right) dy = \int_c^d (b(y) - a(y))\,dy.$$

P.s.m. for $\iint_{\Omega} f(x,y)\,dx\,dy$

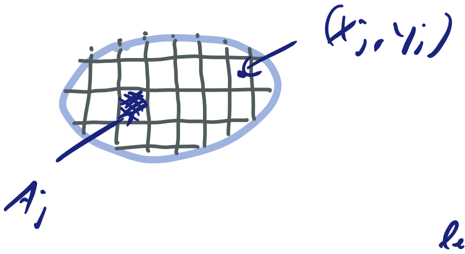


Mål først arealet $\underbrace{\int_{a(y)}^{b(y)} f(x,y)\,dx}_{g(y)}$

Integrér deretter over y til volumet:

$$\int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\,dx \right) dy = \int_c^d g(y)\,dy.$$

Alt 2.



$$\iint_U f(x,y)\,dx\,dy \overset{\text{def.}}{=} \lim_{N\to\infty} \sum_{j=1}^{N} A_j f(x_j,y_j)$$

$$\left(\lim_{N\to\infty} \max_j A_j = 0\right)$$

f Riemann-integrerbar over U dersom høyreleddet har én veldefinert grense for alle valg av A_j, (x_j, y_j) med $\max_j A_j \to 0$.

Teorem (Fubini/Tonelli/Euler, ... : alt 1 og 2 ekvivalente)

La $U \subset \mathbb{R}^2$ være begrenset,

$f \in C(U, \mathbb{R})$ og anta at

$U = \{(x,y) : a(y) \le x \le b(y),\ c \le y \le d\}$

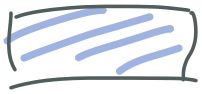
eller

$U = \{(x,y) : a \le x \le b,\ c(x) \le y \le d(x)\}$

a, b, c, d stykkevis kont.

Da er:
$$\iint_U f(x,y)\,dx\,dy = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\,dx \right) dy$$
$$= \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,dy \right) dx.$$

Korollar Dersom $U = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$



og $f(x,y) = h(x)g(y)$,

er
$$\iint_U f(x,y)\,dx\,dy \overset{\text{Tonelli:}}{=} \int_a^b \left(\int_c^d \underset{\text{konst i } y.}{h(x)}\, g(y)\,dy \right) dx$$
$$= \int_a^b h(x) \left(\underbrace{\int_c^d g(y)\,dy}_{\text{konst.}} \right) dx = \left(\int_a^b h(x)\,dx \right)\left(\int_c^d g(y)\,dy \right).$$

Egenskaper til integraler:

(i) Integraler er lineære:

$$\iint (\lambda f + \mu g)\,dA = \lambda \iint f\,dA + \mu \iint g\,dA$$

$\mu, \lambda \in \mathbb{R}$, $f, g : \mathbb{R}^2 \to \mathbb{R}$

$$\boxed{\begin{aligned} &T(\lambda u + \mu v) \\ &= \lambda T(u) + \mu T(v) \end{aligned}}$$

(ii) Int. er additive over disjunkte mengder:

$$\iint_{U \cup V} f\,dA = \iint_{U} f\,dA + \iint_{V} f\,dA \quad \text{dersom} \quad U \cap V = \emptyset.$$

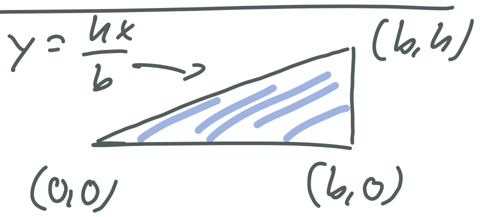
(iii) Int. er monotone operatorer:

$$\iint_U f\,dA \le \iint_U g\,dA \quad \text{dersom} \quad f \le g.$$

(iv) Δ-ulikh. + begrensning:

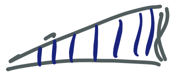
$$\left| \iint_U f\,dA \right| \overset{\Delta\text{-ulikh.}}{\le} \iint_U |f|\,dA \le A(U) \sup_U |f|$$

Eks. Areal av trekant



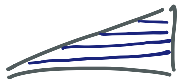
To muligheter:

$$T_1 = \{(x,y) : 0 \le x \le b,\ 0 \le y \le \tfrac{hx}{b}\}$$



og

$$T_2 = \{(x,y) : \tfrac{by}{h} \le x \le b,\ 0 \le y \le h\}$$



$$A(T_1) = \int_0^b \left(\int_0^{\frac{hx}{b}} dy \right) dx = \int_0^b \frac{hx}{b}\, dx = \frac{h}{b} \frac{x^2}{2} \Big|_0^b$$

$$= \frac{bh}{2}. \quad \text{ok!}$$

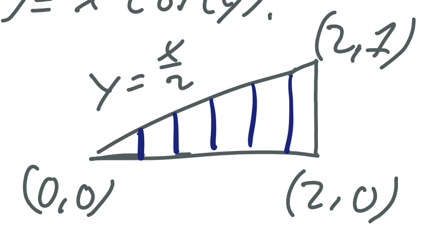
$$A(T_2) = \int_0^h \left(\int_{\frac{by}{h}}^b dx \right) dy = \int_0^h \left(b - \frac{by}{h}\right) dy$$

$$= b \left[y - \frac{y^2}{2h} \right]_0^h = b\left(h - \frac{h^2}{2h}\right) = \frac{bh}{2}.$$

---

ok! Finnes ikke én måte, men uendelig mange å parametrisere på.

Eks Nå, med funksjon $f(x,y) = x\cos(y)$.



$$\begin{cases} 0 \le x \le 2 \\ 0 \le y \le \frac{x}{2} \end{cases}$$

$$\iint_T f\, dA \overset{\text{Tonelli}}{=} \int_0^2 \int_0^{x/2} x\cos(y)\, dy\, dx = \int_0^2 x \underbrace{\int_0^{x/2} \cos(y)\, dy}_{}\, dx$$

$$\sin(y)\Big|_0^{x/2}$$

$$= \int_0^2 \underset{\downarrow}{x} \underset{\uparrow}{\sin}\left(\tfrac{x}{2}\right) dx = \left[-2x\cos\left(\tfrac{x}{2}\right)\right]_0^2 + 2\int_0^2 \cos\left(\tfrac{x}{2}\right) dx$$

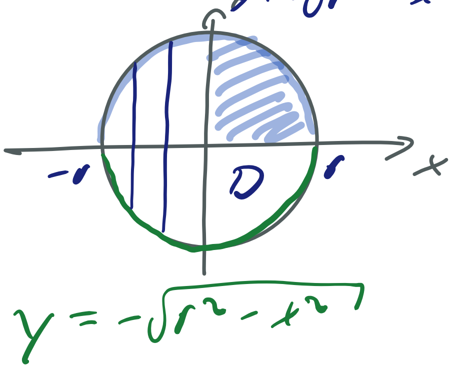
$$= -4\cos(1) + 4\sin\left(\tfrac{x}{2}\right)\Big|_0^2$$

$$= -4\cos(1) + 4\sin(1).$$

\#

Eks Areal av disk med radius $r>0$.

$\{(x,y) : x^2+y^2 \leq r^2\}$



parameterisering:

$$\begin{cases} -r \leq x \leq r \\ -(r^2-x^2)^{\frac{1}{2}} \leq y \leq (r^2-x^2)^{\frac{1}{2}} \end{cases}$$

$$\iint_D dA \overset{\text{Tonelli}}{=} \int_{-r}^{r}\left(\int_{-\sqrt{\ }}^{\sqrt{\ }} dy\right) dx \overset{\text{sym.}}{=} \underline{4\int_0^r \int_0^{\sqrt{r^2-x^2}} dy\, dx}$$

$$= 4\int_0^r \sqrt{r^2-\underline{x^2}}\ \underline{dx} = \left[\begin{aligned} &\underline{x = r\sin(\theta)} \\ &dx = \underline{r\cos(\theta)d\theta} \end{aligned}\right]$$

$\cos(\theta) \geq 0$

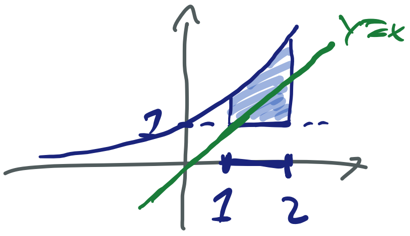
$$= 4\int_0^{\pi/2} \underline{\sqrt{r^2 - r^2\sin^2\theta}}\ \underline{r\cos(\theta)d\theta}$$

$$\sqrt{r^2(\underbrace{1-\sin^2(\theta)}_{\cos^2(\theta)})} = \sqrt{r^2\cos^2(\theta)} = \cancel{\pm}\, r\cos(\theta)$$

$$= 4r^2\int_0^{\pi/2} \underbrace{\cos^2(\theta)}_{\frac{1+\cos(2\theta)}{2}}\, d\theta = \frac{4r^2}{2}\left[\theta + \cancel{\frac{\sin(2\theta)}{2}}\right]_0^{\pi/2}$$

$$= \pi r^2.$$

Eks omps. 5 7/8 2000



Beregn $\iint_R |x-y|\,dA$

for $R = \{(x,y) : 1 \le x \le 2,\ 1 \le y \le e^x\}$ $\boxed{\begin{matrix}1\le x\le e^x\\ x\in[1,2]\end{matrix}}$

$$|x-y| = \begin{cases} x-y, & x \ge y. \\ y-x, & y > x. \end{cases}$$

$$\iint_R |x-y|\,dA = \overset{I_1}{\iint_{R\cap\{x\ge y\}} (x-y)\,dA} + \overset{I_2}{\iint_{R\cap\{y>x\}} (y-x)\,dA}$$

$$I_1 = \int_1^2 \left(\int_1^x (x-y)\,dy\right) dx = \int_1^2 \left[x(x-1) - \underbrace{\frac{y^2}{2}\Big|_1^x}_{-\frac{x^2}{2}+\frac{1}{2}}\right] dx$$

$$= \int_1^2 \left(\frac{x^2}{2} - x + \frac{1}{2}\right) \overset{!}{=} \frac{1}{6}.$$

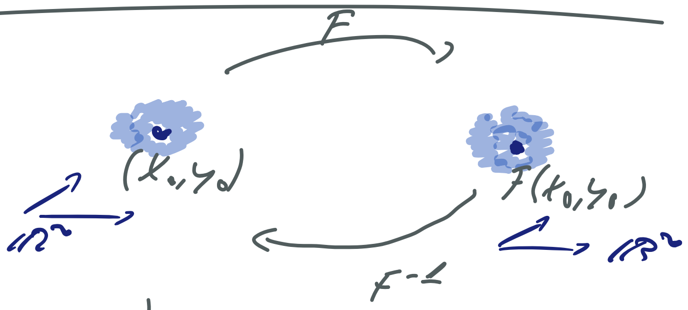
$$I_2 = \int_1^2 \left(\int_x^{e^x} (y-x)\, dy \right) dx = \int_1^2 \left[\frac{y^2}{2} \Big|_x^{e^x} - (e^x - x)x \right] dx$$

$$= \int_1^2 \left(\frac{e^{2x}}{2} - \frac{x^2}{2} - xe^x + x^2 \right) dx = \frac{e^4}{4} - \frac{5e^2}{4} + \frac{7}{6}$$

$$\Longrightarrow A(R) = \frac{e^2}{4}(e^2 - 5) + \frac{4}{3}.$$

◇

Problem fra øving



$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

$$F(x,y) = (\underbrace{x\cos(y)}_{u}, \underbrace{x\sin(y)}_{v})$$

Fikser (x_0, y_0) med $x_0 \neq 0$.

(i) Vis $\exists F^{-1} \in C^1$ nært $F(x_0, y_0)$

(ii) Beregn $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$ i $F(x_0,y_0) = (u_0, v_0)$.

Løsn. Omvendte f.setn. F C^1 med $\det(DF(x_0,y_0)) \neq 0$

$\Longrightarrow \exists F^{-1}$ C^1 nært $F(x_0,y_0) = (u_0, v_0)$.

$$\det(DF(x,y)) = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \det \begin{bmatrix} \cos(y) & -\underline{x}\sin(y) \\ \sin(y) & \underline{x}\cos(y) \end{bmatrix}$$

$$= \underline{x}\left(\cos^2(y) + \sin^2(y)\right) = x \neq 0 \iff x \neq 0.$$

Ja: $\exists\, \varepsilon > 0$ og $F^{-1}: B_\varepsilon(u_0, v_0) \longrightarrow W(x_0, y_0)$.

åpen omegn

(ii) $D(F^{-1})(u_0, v_0) = [DF(x_0, y_0)]^{-1}$

$$F(x,y) = (\underbrace{x\cos(y)}_{u}, \underbrace{x\sin(y)}_{v})$$

Implisitt derivasjon

$$\boxed{u^2 + v^2 = x^2} \iff x = \pm\sqrt{u^2+v^2}$$

$$\frac{\partial}{\partial u}: 2u = 2x\frac{\partial x}{\partial u} \iff \boxed{\frac{\partial x}{\partial u} = \frac{u}{x} = \frac{u}{\pm\sqrt{u^2+v^2}}}$$

↑ bestemt av tegnet til x_0.

$$\frac{\partial}{\partial v}: 2v = 2x\frac{\partial x}{\partial v} \iff \boxed{\frac{\partial x}{\partial v} = \frac{v}{\pm\sqrt{u^2+v^2}}}$$

Kan løses eksplisitt, men kun lokalt:

$$x = \underset{\uparrow\,!}{\pm}\sqrt{u^2+v^2}$$

$$\tan(y) = \frac{v}{u} \iff y = \arctan\Big(\underset{\nearrow}{\frac{v}{u}}\Big) + \underset{\nearrow}{k\pi}$$

må unngå $u=0$!

<u>Bevis omvendte f.teorem</u> (M. Spivak, 1940–2020)

La $\Lambda = DF(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ med $\det(\Lambda) \neq 0$.
invertibel matrise $n \times n$.

$$D(\underbrace{\Lambda^{-1}F}_{G})(x_0) = \Lambda^{-1}DF(x_0) = \Lambda^{-1}\Lambda = Id$$

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

<u>Dersom</u> $\exists G^{-1}$; så $\exists F^{-1} = G^{-1}\circ\Lambda^{-1}$:

$$F\circ F^{-1} = \Lambda \underbrace{G\circ G^{-1}}_{id}\circ\Lambda^{-1} = \Lambda\Lambda^{-1} = id.$$

Så vi viser setningen for G ($\rightsquigarrow F$).

med $DG(x_0) = id = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$.

$$\boxed{D(F^{-1}) = D(G^{-1}\circ\Lambda^{-1})}$$

G C^1: $G(x_0+h) = G(x_0) + DG(x_0)h + |h|\underbrace{\varepsilon(h)}_{\to 0}$

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\begin{bmatrix} h \end{bmatrix}$$ da $h \to 0$

$$\underset{\exists\, \delta > 0}{\Longrightarrow} \quad \frac{|G(x_0+h) - G(x_0)|}{|h|} \neq 0 \quad \text{når} \quad 0 < |h| < \delta.$$

$\Longrightarrow \quad G(x) \neq G(x_0)$ for x nært x_0; $x \in B_\delta(x_0)$

$B_\delta(x_0) = \{x : |x - x_0| < \delta\}$.

G C^1 $\Longleftrightarrow$ DG kont $\Longrightarrow$ $DG(x) = Id + \underbrace{A(x)}_{\to 0}$

da $x \to x_0$.

Nå: Viser at G er injektiv på $B_\delta(x_0)$.

Betrakt $\quad \big| (G(x) - x) - (G(y) - y) \big|$

$$\overset{\Delta\text{-ulikh.}}{\leq} \sum_{j=1}^{n} \big| (G_j(x) - x_j) - (G_j(y) - y_j) \big|$$

$$= \sum_{j=1}^{n} \Big| \underbrace{D(G_j(x) - x_j)}_{1 + A_j(x)|_{x=c} - 1} \Big|_{x=c} \Big| \; |x - y|$$

$$\leq \underbrace{\sum_{j=1}^{n} |A_j(c)|}_{\leq \frac{1}{2}} \; |x - y|$$

dersom x, y nært x_0, dvs velg δ lit liten

Men har også: $|G(x)-x-(G(y)-y)|$

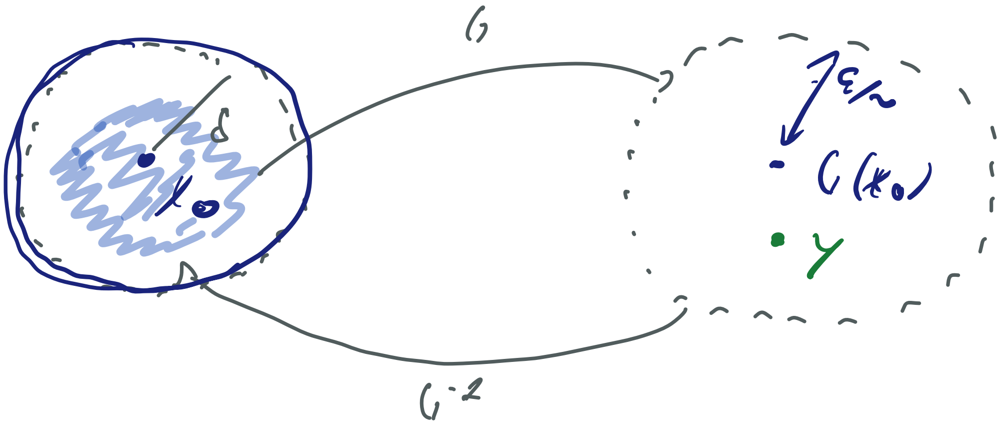
$\geq \underline{|x-y| - |G(x)-G(y)|}$

Δ-ulikh.

$\Longrightarrow \boxed{\frac{1}{2}|x-y| \leq |G(x)-G(y)|}$

G injektiv på $B_\delta(x_0)$: $x \neq y \Longrightarrow G(x) \neq G(y)$.

- G surjektiv? (dvs på hva?)



$x \longmapsto |G(x)-G(x_0)|$ kont,

$\{|x-x_0|=\delta\}$ kompakt

Ekstr. verdi.setn. $\Longrightarrow$ $\exists \min |G(x)-G(x_0)| > 0$

$$\underbrace{x \in \partial B_\delta(x_0)}_{\varepsilon}$$

Betrakt $B_{\varepsilon/2}(G(x_0))$

unik

Vil vise $\forall y \in B_{\varepsilon/2}(G(x_0))\ \exists!\ x \in B_\delta(x_0)$;

$G(x) = y$.

Hvordan: Fiksér y og betrakt avstanden $h(x) = |y - G(x)|^2 = \sum_{j=1}^{n} (y_j - G_j(x))^2$

h kont. på $\overline{B_\delta(x_0)} \Longrightarrow \exists \min\limits_{\overline{B_\delta(x_0)}} h$

$x \in \partial B_\delta$?

$$|y - G(x)| \overset{\Delta\text{-ulikh.}}{\geq} \underbrace{|G(x) - G(x_0)|}_{\geq \varepsilon} - \underbrace{|G(x_0) - y|}_{< \varepsilon/2}$$

$$> \varepsilon/2 > |y - G(x_0)|,$$

Så $x \in \partial B_\delta$ kan ikke gi minimum.

Så hva for $x_y \in B_\delta$: $\boxed{Dh(x_y) = 0.}$

$$h(x) = (G(x) - y) \cdot (G(x) - y)$$

$$Dh(x) = 2\,(G(x) - y) \cdot \underbrace{DG(x)}_{Id + \underset{\to 0}{A(x)} \text{ da } x \to x_0} = 0$$

$\Rightarrow$ $y = G(x)$ i min!

- $\forall y \in B_{\varepsilon/2}(G(x_0))$ har vi funnet én $x \in B_\delta(x_0)$ i $G(x) = y$.
- Kont til G^{-1}.

$$\boxed{\frac{1}{2}|x-y| \leq |G(x) - G(y)|}$$

#

14.4. Variabelsubstitusjon i dobbeltintegraler

Teorem Anta at $\iint_U dA$ eksisterer, $U \subset \mathbb{R}^2$.

La $\Phi : U \to \mathbb{R}^n$ være C^1 med

$\underbrace{\det(D\Phi)}_{\text{Jakobidet.}} \neq 0$ på $\bar{U}$. Da gjelder:

$$\boxed{\iint_{\Phi(\bar U)} f \, dA = \iint_{\bar U} f \circ \Phi \, \underbrace{|D\Phi|}_{|\det(D\Phi)|} \, dA}$$

for f integrerbar.

Eks. (i $\mathbb{R}$!) $U = [0,1]$, $\Phi(x) = 5x+2$

$\Phi(U) = [2,7]$

$D\Phi = \frac{d\Phi}{dx} = 5 \neq 0$ (obs! $\exists \Phi^{-1} \in C^1$ (lokalt))

$$\boxed{\int_{[2,7]} f(y)\, dy = \int_{[0,1]} f(5x+2) \overset{|D\Phi|}{5} \, dx}$$

---

obs! I 1D dukker en orientering opp,

som den i $dy = -dx$ ved bytte $y = -x$.

Den er ikke synlig i teoremet eller $\square$,

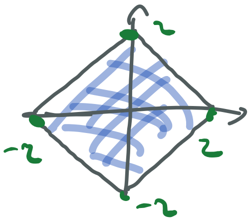
fordi vi bruker absoluttbeløp og mengder.

Sml. • $x = -y$, $\left|\frac{dx}{dy}\right| = 1$: $\int_{[-1,0]} f(y)\,dy = \int_{[0,1]} f(-x)\,1\,dx$

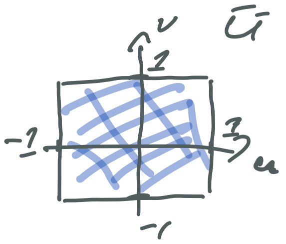
• Med orientering: $\int_{-1}^{0} f(y)\,dy = -\int_{1}^{0} f(-x)\,dx = \int_{0}^{1} f(-x)\,dx.$

Eks i $\mathbb{R}^2$: skalert rotasjon

Gitt $\underbrace{V}_{\Phi(U)} = \left\{-1 \le \frac{x+y}{2} \le 1,\ -1 \le \frac{x-y}{2} \le 1\right\}$



Forenkle: La $\begin{cases} u = \frac{x+y}{2} \\ v = \frac{x-y}{2} \end{cases} \Longleftrightarrow$



$\begin{cases} x = u+v \\ y = u-v. \end{cases}$ $\quad \Phi : (u,v) \mapsto (x,y)$

$$|D\Phi| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = |-2| = 2 \neq 0$$

Så $\iint_V dx\,dy = 2 \iint_U du\,dv.$

Eks 1. Polarkoord.

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

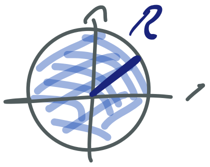
$$\Phi : (r,\theta) \mapsto (x,y)$$

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

$$|D\Phi| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= |r| \neq 0$$ utenfor origo.

$$\bar{V} = \{x^2+y^2 \le R^2\}$$



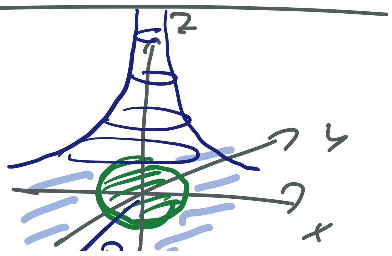
$$\bar{U} = \Phi^{-1}(\bar{V}) = \{0 \le \theta < 2\pi,\ 0 < r \le R\}$$

$$\iint\limits_{V:\, x^2+y^2 \le R^2} dx\,dy = \iint\limits_{\bar{U}} \underset{\uparrow |D\Phi|}{r}\,dr\,d\theta = \int_0^{2\pi}\int_0^R r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^R r\,dr = 2\pi \frac{r^2}{2}\Big|_0^R = \underline{\underline{\pi R^2}}$$

#

Eks 3 Uegentlige integraler



!

$$\iint_{1\le x^2+y^2\le R^2} \frac{dx\,dy}{x^2+y^2} = \int_0^{2\pi}\int_1^R \frac{r\,dr\,d\theta}{r^2}$$

$$\boxed{\int_1^\infty \frac{dx}{x} = \infty}$$

$$= \int_0^{2\pi} d\theta \int_1^R \frac{dr}{r} = 2\pi \ln(r)\Big|_1^R = 2\pi \ln(R) \to \infty$$

da $R \to \infty$.

Så integralet $\iint_{1\le x^2+y^2} \frac{dx\,dy}{x^2+y^2}$ divergent.

Eks 4 $$\iint_{1\le x^2+y^2\le R^2} \frac{dx\,dy}{(x^2+y^2)^{3/2}} = \int_0^{2\pi}\int_1^R \frac{r\,dr\,d\theta}{r^3}$$

$$= 2\pi\left[-\frac{1}{r}\right]_1^R = 2\pi\left(1-\frac{1}{R}\right) \xrightarrow{R\to\infty} 2\pi.$$

Integralet konv. da $R \to \infty$.

Nå: Trippelintegraler 14.5, 14.6 + 10.6

$$\boxed{\iiint_U dV = \iiint_U dx\,dy\,dz \text{ måler volumet til } U.}$$

- Samme def. (Riemannsummer) som for $\mathbb{R}$ og $\mathbb{R}^2$, men med små legemer/kuber.

- $\iiint_{E} f(x,y,z)\,dx\,dy\,dz$ måler oftest en størrelse over et legeme, f.eks. densitet, intensitet, varme, ...   '$\iiint_{E}$ densitet = vekt'

- Som tidligere, kan trippelintegraler beregnes ved 'iterert integrasjon':

Fubini/Tonelli: f, g, h kontinuerlige,

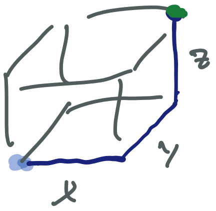
$$E = \{g(x,y) \leq z \leq h(x,y) : (x,y) \in D\}$$

$$\Longrightarrow \quad \iiint_{E} f \, dV = \iint_{D} \left(\int_{g(x,y)}^{h(x,y)} f(x,y,\cdot)\,dz \right) dA$$

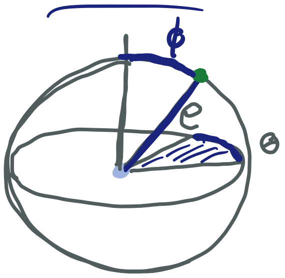
Vanlige variabelrepresentasjoner i $\mathbb{R}^3$:

sylindriske og sfæriske koord.
(kulekoord.)

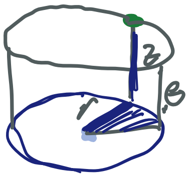
Kartesiske

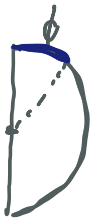


Sfæriske



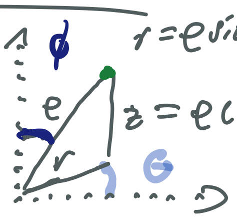
Sylinder-





$0 \le \phi \le \pi$

$$\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = z = \rho\cos\phi \end{cases}$$



$r = \rho\sin\phi$

$z = \rho\cos\phi$

$r = \sqrt{x^2+y^2}$

$\rho = \sqrt{x^2+y^2+z^2}$

$$|D\Phi|_{syl.} = \begin{array}{c} \begin{array}{ccc} \partial_r & \partial_\theta & \partial_z \end{array} \\ \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \end{array} = r.$$

$$\Rightarrow \boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz} \quad (\text{polark.} + z)$$

$$|D\Phi| \underset{\text{stor.}}{=} \left|\begin{array}{ccc} \partial\rho & \partial\phi & \partial\theta \\ \cos\theta\sin\phi & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{array}\right|$$

$$= 0 + \underline{\rho^2\cos^2\phi\sin\phi\cos^2\theta} + \underline{\rho^2\sin\phi\sin^2\phi\sin^2\theta}$$

$$- \left[\underline{-\rho^2\sin^3\phi\cos^2\theta} + 0 - \underline{\rho^2\sin\phi\cos^2\phi\sin^2\theta}\right]$$

$$= \rho^2\left[\underline{\sin^2\phi\sin\phi} + \underline{\cos^2\phi\sin\phi}\right] = \underline{\underline{\rho^2\sin\phi}}$$

obs $0 \le \phi \le \pi$
$\sin\phi \ge 0.$

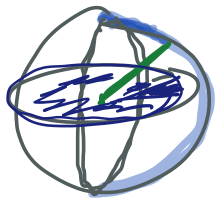
$$\Rightarrow \boxed{dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

Eks. volum av en kule med radius $R > 0$.

$$\iiint_{x^2+y^2+z^2\le R^2} dx\,dy\,dz = \underset{\substack{\uparrow \\ \text{stor.}}}{=} \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{R}\rho^2\,d\rho\right)$$

$$= 2\pi\underbrace{\left[-\cos\phi\right]_0^{\pi}}_{2}\left.\frac{\rho^3}{3}\right|_0^R = \frac{4\pi}{3}R^3.$$



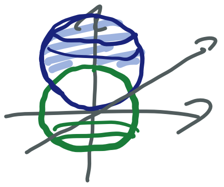
$0 \le \phi \le \pi$

$0 \le \theta < 2\pi$

$0 \le \rho \le R$

Eks: integral over $B_1(0,0,1) \setminus B_1(0,0,0)$

$V = \{(x,y,z) : x^2+y^2+(z-1)^2 < 1$
og $x^2+y^2+z^2 \geq 1\}$



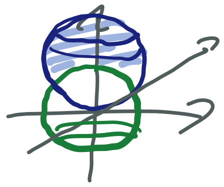
$0 \leq \theta < 2\pi$

$S_{z=1} : x^2+y^2+(z-1)^2 = 1$

$\iff \rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + (\rho\cos\phi \cancel{-1})^2 = \cancel{1}.$

$\iff \rho^2 (\underbrace{\sin^2\phi + \cos^2\phi}_{1}) = 2\rho\cos\phi$

$\iff \begin{cases} \text{~~}\rho = 0 \text{ eller~~} \\ \boxed{\rho = 2\cos\phi} \text{ (inkl. } \rho = 0) \end{cases}$

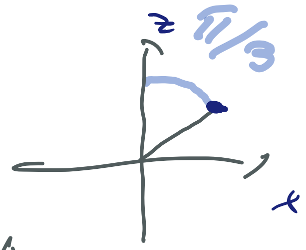


$S_{z=0} : x^2+y^2+z^2 = 1 \iff \boxed{\rho = 1.}$

Så: $1 \leq \rho < 2\cos\phi$
$0 \leq \phi < \frac{\pi}{3}$

Finn skjæringen!

Skjæring: $\cos\phi = \frac{1}{2} \iff \phi = \frac{\pi}{3}.$



Integral: $\int_0^{2\pi} \int_0^{\pi/3} \int_1^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$= 2\pi \int^{\pi/3} \sin\phi \left[\frac{\rho^3}{3} \right]^{2\cos\phi} d\phi$

$0 \qquad\qquad 2$

$$= \frac{2\pi}{3}\int_0^{\pi/3}\left(8\underline{\sin\phi}\cos^3\phi - \sin\phi\right)d\phi$$

$$= \frac{2\pi}{3}\left[-2\cos^4\phi + \cos\phi\right]_0^{\pi/3}$$

$$= \frac{2\pi}{3}\left[-\frac{1}{8}+\frac{1}{2}+2-1\right] = \frac{11}{12}\pi.$$

#

Eks. opps. ⑤ 14/5 2002

Beregn volumet til en kirke med vegger $x^2+y^2=(18)^2$, tak $z = 20 - \frac{x^2}{25} + \frac{y}{2}$ for $x^2+y^2 \le 400$ og gulv ved $z=0$.

Løsn $D = \{(x,y,z): 0 \le \overset{r^2}{x^2+y^2} \le (18)^2,$
$0 \le z \le 20 - \frac{x^2}{25} + \frac{y}{2}\}$ $-\pi \le \theta \le \pi$

$$\iiint_D dV = \iint_{\underline{x^2+y^2\le 18^2}} \left(\underline{\int_0^{20-\frac{x^2}{25}+\frac{y}{2}} dz}\right)dx\,dy = \begin{bmatrix}\text{Sylinder}\\ \text{koord.}\end{bmatrix}$$

$$= \int_{\boxed{-\pi}}^{\boxed{\pi}}\int_0^{18}\Big(\underline{20} - \underbrace{\frac{r^2\cos^2\theta}{25}}_{\frac{2\pi 18^2}{2\cdot 25}} + \underbrace{\frac{r\sin\theta}{2}}_{=0}\Big)\underline{\underline{r}}\,dr\,d\theta = (*)$$

π

sin odde $\Rightarrow \int_{-\pi}^{\pi} \sin\theta \, d\theta = 0$

cos² like med $\int_{-\pi}^{\pi} \cos^2\theta \, d\theta = \int_{-\pi}^{\pi} \frac{1+\cos(2\theta)}{2} \, d\theta = \frac{1}{2} \cdot 2\pi$

$$(*) = \cancel{2}\pi \left[\underline{\frac{20 r^2}{\cancel{2}}} - \frac{r^4}{4 \cdot \cancel{2} \cdot 25} \right]_0^{18}$$

$$= \pi \left[20 \cdot 18^2 - \frac{18^4}{100} \right] = \underline{\underline{5430{,}24\pi}}.$$

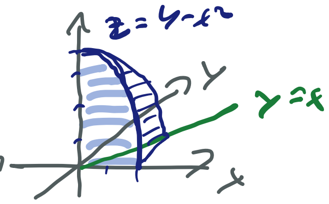
Volumet.

Eks. opps. 4 8.8 2001

Beregn $\int_0^2 \int_0^{4-x^2} \int_0^x \underline{\frac{\sin(2z)}{z-4}} \, dy \, dz \, dx = (**)$

Løsn. $0 \leq \underline{y} \leq x$

$$\begin{cases} 0 \leq z \leq \underline{4-x^2} \\ 0 \leq x \leq 2 \end{cases} \Longleftrightarrow \begin{matrix} 0 \leq z \leq 4 \\ 0 \leq x \leq \sqrt{4-z} \end{matrix}$$



Kaster om
int. rekkefølgen!

1. $\int_0^x \frac{\sin(2z)}{z-4} \, dy = \underline{\frac{x \sin(2z)}{z-4}}$

2. Reparametriser:

$$(*) = \int_0^4 \int_0^{\sqrt{4-z}} \frac{x \sin(2z)}{z-4}\, dx\, dz$$

$$\frac{1}{z-4} \cdot \frac{4-z}{2}$$

$$= \int_0^4 \frac{\sin(2z)}{\cancel{z-4}} \underbrace{\left(\int_0^{\sqrt{4-z}} x\, dx \right)}_{\frac{x^2}{2}\Big|_0^{\sqrt{4-z}} = \cancel{\frac{4-z}{2}}} dz = -\frac{1}{2} \int_0^4 \sin(2z)\, dz$$

obs! tegn!

$$= \frac{1}{4}(\cos(8) - 1).$$

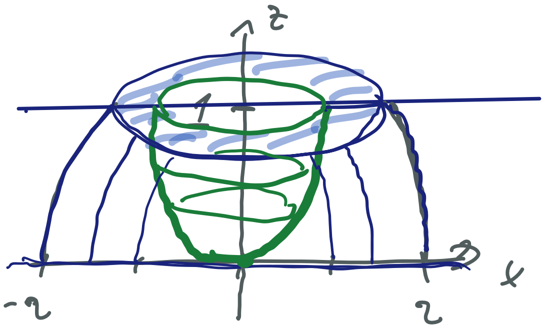
Eks. oppg. 3 19.12 2006

R avgrenset av $x^2+y^2+z^2=4$, $z = x^2+y^2$,

kuleskall paraboloide

$z=0$ og $z=1$.

plan plan



a) Skisser R.

Beregn V(R).

$$R = \{ 0 \le z \le 1,\ z \le x^2+y^2 \le 4-z^2 \}$$

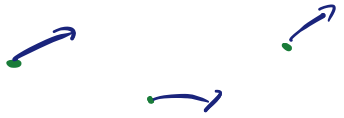
$$\iiint_R dV = \left[\begin{matrix} \text{syl.} \\ \text{koord.} \end{matrix} \right] = \int_0^{2\pi} \int_0^1 \int_{\sqrt{z}}^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$$

$$= \cancel{2}\pi \int_0^1 \left[\frac{r^2}{\cancel{2}}\right]_{\sqrt{z}}^{\sqrt{4-z^2}} dz \underset{\substack{\uparrow \\ 4-z^2 \geq 0 \\ 0 \leq z \leq 1}}{=} \pi \int_0^1 ((4-z^2)-z)\,dz = \frac{19\pi}{6}.$$

15.1 - 15.2 Vektor- og skalarfelt

Def. • Et vektorfelt er en funksjon/avbildning $U \subset \mathbb{R}^n \to \mathbb{R}^n$.

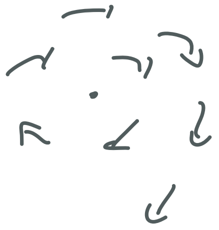
Til hvert punkt i $\mathbb{R}^n$



tildeler vi en vektor av samme dimensjon.

Typisk eksempel:

Værkartet



• Et skalarfelt er en avbildning $U \subset \mathbb{R}^n \to \mathbb{R}$.

Det vanligste eksemplet på vektorfelt er et gradientfelt.

Def. $F : U \overset{\text{åpen}}{\subset} \mathbb{R}^n \to \mathbb{R}^n$ gradientfelt $\overset{\text{def.}}{\iff}$

$F = \nabla f$, $\underset{\wedge}{f} : U \subset \mathbb{R}^n \to \mathbb{R}$

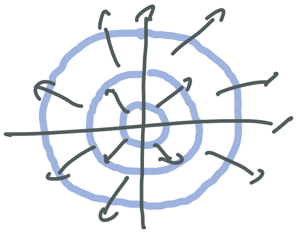
'skalarfelt

Eks. $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2+y^2$

$F=\nabla f:(x,y)\mapsto(2x,2y)$ er et gradientfelt.

Def Et vektorfelt F er *konservativt* dersom det finnes en $\phi:\mathbb{R}^n\to\mathbb{R}$; $F=\nabla\phi$.

Funksjonen ϕ kalles *potential til F* (unik opp til en konstant).



Når finnes en potential?

- *Dersom* ϕ finnes, må $F=\left(\frac{\partial\phi}{\partial x_1},\dots,\frac{\partial\phi}{\partial x_n}\right)$.

Så, gitt F er C^1, blir ϕ C^2, og

$\partial x_i\underbrace{\partial x_j}\phi=\partial x_j\underbrace{\partial x_i}\phi$, $1\le i,j\le n$, dvs

$$\boxed{\frac{\partial F_j}{\partial x_i}=\frac{\partial F_i}{\partial x_j}}$$

er et *nødvendig* vilkår for eksistens av en potential $\phi\in C^2$.

Hva er bra med konservative felt?

Som å finne anti-deriverte i integraler.

Først: kurveintegraler

15.3 - 15.4 Kurveintegraler av felt

Husk at $\gamma \in C^1(I, \mathbb{R}^n)$ med $\underset{|v|}{|\dot{\gamma}|} \neq 0$ på I,

har lengde $\boxed{\int_\gamma ds = \int_I |\dot{\gamma}(t)|\, dt}$, der

$$s(t) = \int_{t_0}^{t} |\dot{\gamma}(\tau)|\, d\tau, \quad \text{så} \quad \frac{ds}{dt} = |\dot{\gamma}(t)| \neq 0.$$

Def. $I \subset \mathbb{R}$, $U \subset \mathbb{R}^n$, $\gamma \in C^1(I, \mathbb{R}^n)$:

$\gamma(I) \subset \bar{U}$ og $|\dot{\gamma}| \neq 0$ $\forall t \in I$.

Kurveintegralet av et skalarfelt $f \in C(U, \mathbb{R})$

langs kurven γ er:

$$\boxed{\int_\gamma f\, ds = \int_I f(\gamma(t))\, |\dot{\gamma}(t)|\, dt}$$

Eks. oppg. 4(a) 8/8 2005

Beregn buelengde til $C = \left\{\left(\frac{1}{t}, \sqrt{2}t, \frac{t^3}{3}\right) : \frac{1}{2} \le t \le 2\right\}$

Løsn. $\gamma : t \mapsto \left(\frac{1}{t}, \sqrt{2}t, \frac{t^3}{3}\right)$ er C^1 på $(0,\infty)$ $\uparrow !$

og på $\frac{1}{2} \le t \le 2$ med

$$|\dot\gamma(t)| = \left|\left(-\frac{1}{t^2}, \sqrt{2}, t^2\right)\right| = \sqrt{\underbrace{\frac{1}{t^4} + 2 + t^4}_{\left(\frac{1}{t^2}+t^2\right)^2}}$$

$$= \frac{1}{t^2} + t^2 > 0 \text{ på } (0,\infty).$$

Buelengde: $\int_\gamma ds = \int_{1/2}^{2} |\dot\gamma(t)|\,dt = \int_{1/2}^{2} \left(\frac{1}{t^2} + t^2\right) dt$

$$= \left[-\frac{1}{t} + \frac{t^3}{3}\right]_{1/2}^{2} = -\frac{1}{2} + \frac{8}{3} + 2 - \frac{1}{3\cdot 8} = \frac{33}{8}.$$

Eks. oppg. 1 4/8 2008 (modifisert)

$$\gamma : t \mapsto \underbrace{(x,y,z) = \left(t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3}\right)}_{x(t)}, \quad 0 \le t \le 2.$$

Beregn $\int_\gamma \overset{\swarrow}{x}\, ds.$

Løsn. γ glatt med $|\dot{\gamma}(t)| = |(1, \sqrt{2}t, t^2)|$

$$= \sqrt{\underbrace{1+2t^2+t^4}_{(1+t^2)^2}} = 1+t^2 > 0 \quad \forall t \in \mathbb{R}.$$

$\Rightarrow ds = (1+t^2)dt$ <u>buelengdeelementet</u>

$$\int_\gamma x\, ds = \int_0^2 t(1+t^2)dt = \left[\frac{t^2}{2} + \frac{t^4}{4}\right]_0^2 = 6.$$

□

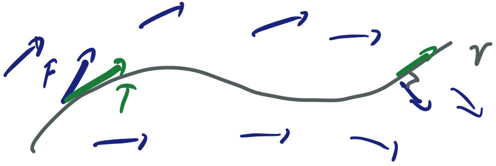
Def. Kurveintegralet av et kontinuerlig vektorfelt

$F : \overset{\text{åpen}}{\bar{U}} \subset \mathbb{R}^n \to \mathbb{R}^n$ langs en glatt kurve

$\gamma : I \to \bar{U}$, er:

$t \mapsto \gamma(t)$

$$\boxed{\int_\gamma \vec{F}\cdot d\vec{r} \overset{\text{def.}}{=} \int_\gamma \vec{F}\cdot \underbrace{\vec{T}\, ds}_{dr} = \int_I F(\gamma(t))\cdot \dot{\gamma}(t)\,dt}$$



$F\cdot T$ projeksjonen av F i retning T, der $|T|=1$.

Merk: $\vec{T}\,ds = \frac{\vec{\dot\gamma}(t)}{\cancel{|\dot\gamma(t)|}}\,\cancel{|\dot\gamma(t)|}\,dt = \vec{\dot\gamma}(t)\,dt.$

Prop $\int_\gamma F\cdot dr$ uavhengig av parameterisering.

Bevis $\int_\gamma F\cdot dr = \int_a^b F(\gamma(t))\cdot\underline{\dot\gamma(t)\,dt} = (*)$

$\gamma(a)=\vec{x_0}$, $\gamma(b)=\vec{y_0}$

La $s(t) = \int_a^t |\dot\gamma(\tau)|\,d\tau$, $s(a)=0$, $s(b)=L(\gamma)$.

$\frac{ds}{dt} = |\dot\gamma(t)|$, $\tilde\gamma(s) = \gamma(t(s))$

$\Rightarrow \boxed{\frac{dr}{dt} = \frac{d\tilde\gamma}{ds}\,\frac{ds}{dt}}$ Variabelsubst. $\Rightarrow \boxed{\dot\gamma(t)\,dt = \tilde\gamma'(s)\,ds}$

$(*) = \int_0^L F(\underbrace{\tilde\gamma(s)}_{\gamma(t(s))})\cdot\tilde\gamma'(s)\,ds$, uavhengig av t.

Buelengdeparameteren er unik. $\Rightarrow$ entydig def.

(uansett param.) //

Teorem $F \in C^1(U, \mathbb{R}^n)$ konservativt,

$\gamma : [a,b] \to U$ glatt,

$$\Longrightarrow \boxed{\int_\gamma F \cdot dr = \phi(\gamma(b)) - \phi(\gamma(a))}$$

avhengig kun av endpunkter

Bevis $$\int_\gamma F \cdot dr = \int_\gamma F \cdot T\, ds = \int_a^b F(\gamma(t)) \cdot \dot{\gamma}(t)\, dt$$

$\nabla \phi = F$

konservativt

$$= \int_a^b \underline{\nabla \phi(\gamma(t)) \cdot \dot{\gamma}(t)}\, dt \underset{\text{kj.reg.}}{\overset{\uparrow}{=}} \int_a^b \frac{d}{dt}\, \phi(\gamma(t))\, dt$$

$$\underset{\text{fund.setn.}}{\overset{\uparrow}{=}} \phi(\gamma(b)) - \phi(\gamma(a)).$$

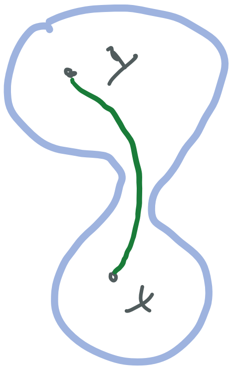
$\boxtimes$

Hvordan vite om ϕ eksisterer? (F konservativt)

Def. (i) $U \subset \mathbb{R}^n$ sammenhengende dersom

$\forall x, y \in \bar{U} \quad \exists \gamma \in C([0,1], \bar{U});$

$$\boxed{\gamma(0) = x, \quad \gamma(1) = y.}$$

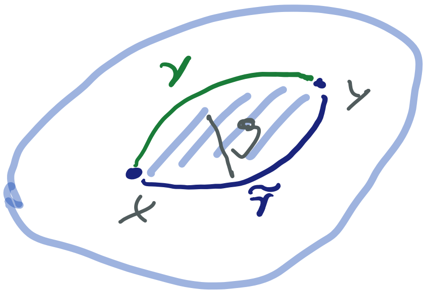


(ii) $\bar{U}$ er enkelt sammenhengende dersom

$\forall \gamma, \tilde{\gamma} \in C([0,1], \bar{U}), \ \forall x, y \in \bar{U},$

$\gamma(0) = \tilde{\gamma}(0) = x,$

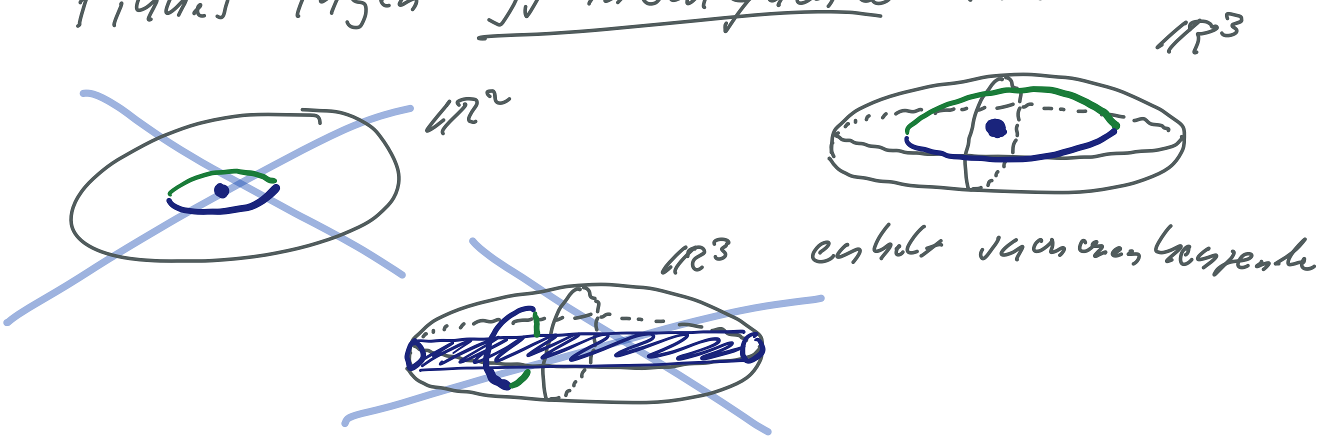
$\gamma(1) = \tilde{\gamma}(1) = y,$



$\exists g : C([0,1]\times[0,1], U), \ (t,\lambda) \to g(t,\lambda);$

$$\boxed{g(t,0) = \gamma(t), \quad g(t,1) = \tilde{\gamma}(t).}$$

Finnes ingen 'gjennomgående hull'



Teorem $U \subset \mathbb{R}^n$ åpen, enkelt sammenhengende, $F \in C^1(U, \mathbb{R}^n)$. Da er følgende vilkår ekvivalente:

(i) F konservativt

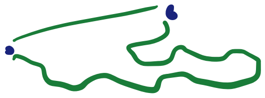
(ii) $\partial_{x_j} F_i = \partial_{x_i} F_j$, $1 \le i, j \le n$

(iii) $\oint_\gamma F \cdot dr = 0$ ∀ lukket kurve γ i U.

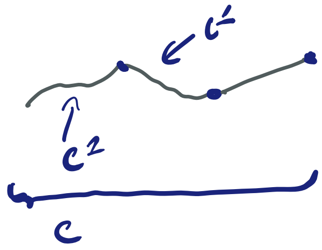
$\gamma(a) = \gamma(b)$.

(iv) $\int_a^b F \cdot dr = \phi(\gamma(b)) - \phi(\gamma(a))$ (Det finnes en ϕ)

'veiuavhengig'



Merk: Tilstrekkelig at γ er stykkevis glatt, dvs



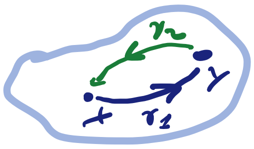
Gjelder alle kurveintegraler!

Bevis (i) ⇒ (ii) allerede bevist

(i) ⇒ (iv) — " —

(iv) ⇒ (iii) Trivielt, fordi $\gamma(a) = \gamma(b)$ for en lukket kurve $\underset{(iv)}{\Rightarrow} \phi(\gamma(a)) - \phi(\gamma(b)) = 0$.

(iii) ⇒ (iv)

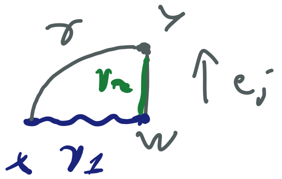


$$\int_{\gamma_1} F\cdot dr = \underbrace{\oint_{\gamma} F\cdot dr}_{0} - \int_{\gamma_2} F\cdot dr = -\int_{\gamma_2} F\cdot dr$$

<u>Så: veiuavhengig</u>

(iv) ⇒ (i) Fikser et punkt $x \in \bar{\Omega}$,
la γ være en kurve fra x til $y \in \bar{\Omega}$.

La $\phi(y) \overset{\text{def.}}{=} \underbrace{\int_{\gamma} F\cdot dr}_{\text{veiuavhengig}}$



$w_j = x_j$

$\frac{\partial}{\partial y_j}$

$$\overset{(iv)}{=} \underbrace{\cancel{\int_{\gamma_1} F\cdot dr}}_{\text{konst. i } e_j} + \underbrace{\int_{\gamma_2} F\cdot dr.}_{\text{varierer i } e_j\text{-retn.}}$$

$x_j < y_j$

$$\gamma_2 : (w_1, w_2, \ldots, \underset{\overset{\uparrow}{j}}{t}, \ldots, w_n), \quad t \in [x_j, y_j]$$

$$\Rightarrow \frac{\partial \phi}{\partial y_j} = \frac{\partial}{\partial y_j} \int_{x_j}^{y_j} F(\gamma(t)) \cdot \underbrace{\dot{\gamma}(t)}_{(0,0,\ldots,1,0,\ldots,0)} dt$$

$$= \frac{\partial}{\partial y_j} \int_{x_j}^{y_j} F_j(\gamma(t))\, dt = \underbrace{F_j(\gamma(y_j))}_{\text{Fund.teo.}} = \underline{\underline{F_j(y)}}$$

Sant for $j = 1, 2, \dots, 4 \Longrightarrow \boxed{\nabla \phi = F.}$

Eks oms. 4 1/8 2002

$$F(x,y,z) = (-y, x, e^{z})$$

$$r : t \longmapsto (\underset{x}{\cos(t)}, \underset{y}{\sin(t)}, \underset{z}{\tfrac{t}{2}}), \quad t \in [0, 4\pi]$$

Beregn $\int_{r} F \cdot T\, ds = \int_{r} F \cdot dr$

Løsn. $\int_{r} F \cdot T\, ds$

$$= \int_0^{4\pi} \underbrace{(-\sin(t), \cos(t), e^{t/2})}_{F(r(t))} \cdot \underbrace{(-\sin(t), \cos(t), \tfrac{1}{2})}_{\dot{r}(t)}\, dt$$

$$T = \frac{\dot{r}}{|\dot{r}|}, \quad ds = |\dot{r}|\, dt$$

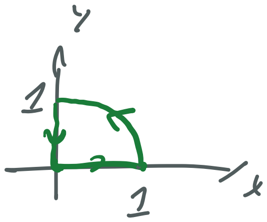
$$= \int_0^{4\pi} \Big(\underbrace{\sin^2(t) + \cos^2(t)}_{1} + \frac{e^{t/2}}{2}\Big) dt = 4\pi + e^{2\pi} - 1.$$

Eks. $F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \to (x^2, xy)$

$\gamma_1 : x \mapsto (x,0),\ x \in [0,1]$ $\rightarrow$

$\gamma_2 : \theta \mapsto (\cos(\theta), \sin(\theta)),\ \theta \in [0, \frac{\pi}{2}]$

$\gamma_3 : y \mapsto (0,y),$ $\underline{y \text{ fra } 1 \text{ til } 0!}$ $\downarrow$



$$\oint_\gamma F\cdot T\,ds = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) F\cdot T\,ds.$$

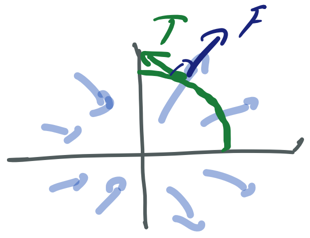
$$\int_{\gamma_1 \rightarrow} F\cdot T\,ds = \int_0^1 \underset{F(\gamma_1(x))}{(x^2,0)}\cdot\underset{\dot\gamma_1(x)}{(1,0)}\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

$$\int_{\gamma_2} F\cdot T\,ds = \int_0^{\pi/2} \underbrace{(\cos^2(\theta), \cos(\theta)\sin(\theta))}_{F(\gamma_2(\theta))}\cdot\underbrace{(-\sin(\theta), \cos(\theta))}_{\dot\gamma_2(\theta)}\,d\theta$$

$$= 0.$$

Hvorfor?

$F(x,y) = (x^2, xy) = x(x,y)$



$$\downarrow \int_{\gamma_3} F\cdot T\,ds$$

$$= \int_1^0 (0,0)\cdot(0,1)\,dy \;!$$

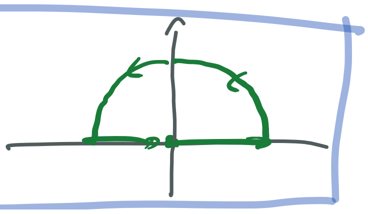
$$= 0$$

Så $\oint_\gamma F\cdot T\,ds = \frac{1}{3}.$

(F ikke konservativt!)

Tjek at

$\int_\gamma F\cdot dr = 0$



Bevis (ii) ⇒ (i), se separat PDF på wikien.

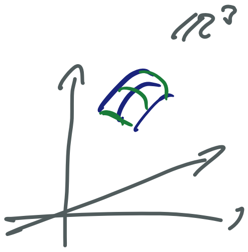
Merk: Når $r(x) = (x, c_2, c_3, \dots, c_n)$ blir

$$F \cdot dr = F \cdot T\,ds = (F_1, F_2, \dots, F_n) \cdot \underbrace{(1, 0, \dots, 0)}_{r'(x)} dx$$
$$= F_1\,dx.$$

Dette motiverer notasjonen:

$$\boxed{\begin{aligned} dr = T\,ds &= F_1\,dx + F_2\,dy + F_3\,dz \quad &\text{i } \mathbb{R}^3 \\ &F_1\,dx + F_2\,dy \quad &\text{i } \mathbb{R}^2 \end{aligned}}$$

15.5 Flater og flateintegraler



Hva er en flate?

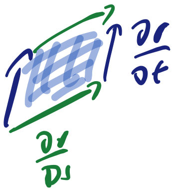
kurver i $\mathbb{R}^2$	flater i $\mathbb{R}^3$	fremstillinger
$y = f(x)$	$z = f(x,y)$	graf
$f(x,y) = c$	$f(x,y,z) = c$	implisitt / nivåmengder (gitt $\nabla f \neq 0$)
$r(t) = (r_1(t), r_2(t))$	?	parametrisk

Def Glatte flater: "To familier av kurver som ikke er parallelle":

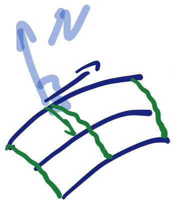
$$(s,t) \longmapsto \vec{r}(s,t) = (r_1(s,t), r_2(s,t), r_3(s,t));$$

$s \mapsto r(s,t)$, $t \mapsto r(s,t)$ glatte kurver, og

$$\underbrace{\left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right|}_{\text{areal til}} \neq 0.$$



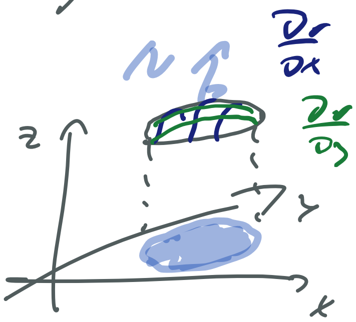
Merk: $\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}$ normal til flaten.



N kont fordi: $\partial_s r$ og $\partial_t r$ er kont.

For en graf $z = f(x,y)$, $f \in C^1$, gir parametriseringen:

$$(x,y) \longmapsto \underbrace{(x, y, f(x,y))}_{r(x,y)}$$



$\frac{\partial r}{\partial x} = (1, 0, f_x(x,y))$: funksjon $\mathbb{R}^2 \to \mathbb{R}^3$

tangent i x-retning

$$\frac{\partial r}{\partial y} = (0, 1, f_y(x,y)) : \mathbb{R}^2 \to \mathbb{R}^3$$

tangent i y-retning

normal: $\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{pmatrix}$

$$= \underline{(-f_x, -f_y, 1)} \neq (0,0,0)$$

For en nivåflate: $f(x,y,z) = c, \quad f \in C^2$

med $|\nabla f| \neq 0 \implies \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ eller $\frac{\partial f}{\partial z} \neq 0$

(i et gitt punkt)

Si $\frac{\partial f}{\partial z} \neq 0 \overset{IFT}{\implies} \exists\, \phi \; C^1 :$

$f(x,y,z) = c \overset{lokalt}{\iff} z = \phi(x,y)$ **lokalt en graf!**

Parametrisering: $(x,y) \mapsto (x, y, \phi(x,y))$

og likt som ovenfor: $\frac{\partial r}{\partial x} = (1, 0, \boxed{\phi_x})$

Men: kan også

$$\frac{\partial r}{\partial y} = (0, 1, \boxed{\phi_y})$$

løse ut ϕ_x, ϕ_y i termer av f:

$$f(x, y, \phi(x,y)) = c \xrightarrow[\partial_y]{\partial_x} \begin{cases} \partial_x f + \partial_z f \, \partial_x \phi = 0 \\ \partial_y f + \partial_z f \, \partial_y \phi = 0 \end{cases}$$

Impl. der.

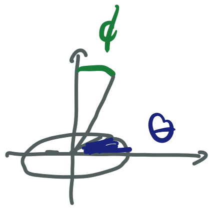
$$\boxed{\partial_x \phi = -\frac{\partial_x f}{\partial_z f}, \qquad \partial_y \phi = -\frac{\partial_y f}{\partial_z f}}$$

$$\Rightarrow \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = (-\phi_x, -\phi_y, 1) = \left(\frac{f_x}{f_z}, \frac{f_y}{f_z}, 1\right)$$

$$= \frac{1}{f_z} \nabla f.$$

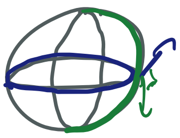
Eks. (i) Enhetssfæren (kuleskall): $x^2+y^2+z^2=1$

a) sfæriske koord. (ϕ, θ)



$$r(\phi, \theta) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$$

$$0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi$$

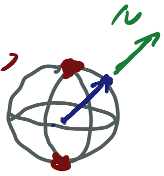


$$\begin{array}{cccc} & e_1 & e_2 & e_3 \\ \frac{\partial r}{\partial \phi} = & (\cos\phi\cos\theta, & \cos\phi\sin\theta, & -\sin\phi) \\ \frac{\partial r}{\partial \theta} = & (-\sin\phi\sin\theta, & \sin\phi\cos\theta, & 0) \end{array}$$

ortogonale!

$$\frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)$$

$$= \underbrace{\sin\phi}_{0 \text{ for } \phi = 0 \text{ og } \phi = \pi} \, r(\phi, \theta) \quad // \quad r \text{ selv!}$$



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b) kartesiske koord. $\underbrace{x^2+y^2+z^2}_{f(x,y,z)} = 1$

Finner en lokal, f.eks.

$$\boxed{\left\{ z = \underbrace{\sqrt{1-x^2-y^2}}_{\phi(x,y)} : 0 \leq x^2+y^2 \leq 1 \right\}} \cup \left\{ z = \underset{\uparrow}{-} \sqrt{\cdots} \right\}$$

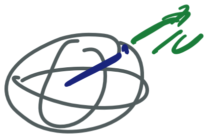
$$(x,y) \overset{r}{\longmapsto} (x, y, \underbrace{\sqrt{1-x^2-y^2}}_{z})$$

$$\frac{\partial r}{\partial x} = \Big(1, 0, \underbrace{\frac{-x}{\sqrt{}}}_{\phi_x}\Big), \quad \frac{\partial r}{\partial y} = \Big(0, 1, \underbrace{\frac{-y}{\sqrt{}}}_{\phi_y}\Big)$$

$$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \frac{\nabla f}{f_z} = \frac{(2x, 2y, 2z)}{2z}$$

$$= \left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\right)$$

Igjen: normalen parallell til r selv.



Mange parameteriseringer mulige!

Øvelse: Plan $\underbrace{Ax + By + Cz}_{f(x,y,z)} = D$,

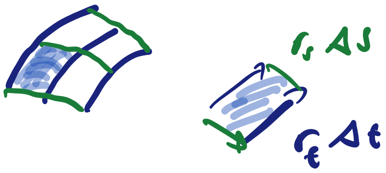
$|(A, B, C)| \neq 0$. Parametriser.

Finn tangentvektorer og normal.

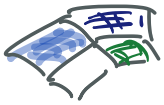
Når kan planet parametriseres i (x,y); i (x,z); i (y,z)?

Nå: ønsker å integrere over en flate S.

Idé:



$$\lim \sum |r_t \times r_s| \Delta s \Delta t$$



Def. (i) Arealet til en flate S gitt ved

$r \in C^1(\bar{U}, \mathbb{R}^3)$, $U \subset \mathbb{R}^2$, med $\left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| \neq 0$,

$s, t \in \bar{U}$, er

$$A(S) \overset{\text{def.}}{=} \iint_{\bar{U}} \underbrace{\left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| ds\, dt}_{d\sigma} = \iint_S d\sigma$$

(ii) Flateintegralet av et skalarfelt $f \in C(\bar{V}, \mathbb{R})$ over S som ovenfor, $S \subset \underset{\text{åpen}}{\bar{V}}$, $V \subset \mathbb{R}^3$, er

$$\iint_S f\, d\sigma = \iint_{\bar{U}} f(r(s,t)) \left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| ds\, dt.$$

Eks Oppg 4 15/8 2009

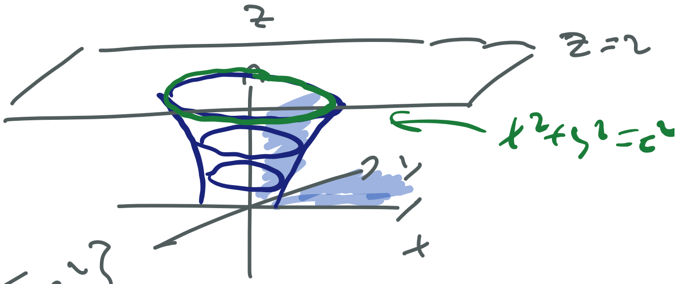
Beregn $A(S)$, der S er delen av flaten $\{z = \ln(x^2+y^2)\}$ i første oktant, under planet $\{z=2\}$.

Løsn. Første oktant: $x, y, z \geq 0$.

$$\Rightarrow \quad 0 \leq z \leq 2. \quad \Rightarrow \quad \boxed{\begin{gathered} 0 \leq \ln(x^2+y^2) \leq 2 \\ x, y \geq 0 \end{gathered}}$$

$$\Leftrightarrow \quad \boxed{\begin{gathered} 1 \leq x^2+y^2 \leq e^2 \\ x, y \geq 0 \end{gathered}}$$

Kartesiske koord.



$$S = \{(x, y, \ln(x^2+y^2)) : x \geq 0, y \geq 0, 1 \leq x^2+y^2 \leq e^2\}$$

$$\left|\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y}\right| = \left| \det \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \frac{2x}{x^2+y^2} \\ 0 & 1 & \frac{2y}{x^2+y^2} \end{pmatrix} \right|$$

$$= \Big\| \underbrace{\left(-\frac{2x}{x^2+y^2}, -\frac{2y}{x^2+y^2}, 1\right)}_{\text{kont på } S!} \Big\| = \sqrt{\frac{4(x^2+y^2)}{(x^2+y^2)^2}+1}.$$

pol. koord. (a,θ)

!

$$d\sigma = \sqrt{\frac{4}{x^2+y^2}+1}\,dx\,dy = \sqrt{1+\frac{4}{a^2}}\,a\,da\,d\theta$$

$$\boxed{\begin{array}{l}(x,y)\leftrightarrow(r,\theta)\\ dx\,dy\\ = r\,dr\,d\theta\end{array}}$$

Så: $$A(S) = \iint_S d\sigma = \int_0^{\frac{\pi}{2}}\int_1^e \sqrt{1+\frac{4}{a^2}}\,a\,da\,d\theta$$

$1 \le a \le e$

$0 \le \theta \le \frac{\pi}{2}$

$$= \frac{\pi}{2}\int_1^e \sqrt{4+a^2}\,da = \dots \text{ (hyperbolsk var.bytte)}$$

Merk: Hvis vi bruker sylinderkoord direkte: $r(a,\theta) = (a\cos(\theta), a\sin(\theta), \ln(a^2))$

$$\Rightarrow \left|\frac{\partial r}{\partial a}\times\frac{\partial r}{\partial \theta}\right| = \sqrt{4+a^2}$$

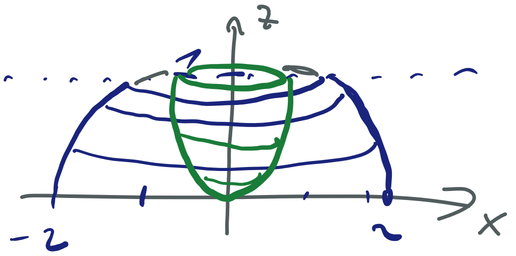
$$d\sigma = \sqrt{4+a^2}\,da\,d\theta$$

Eks opps 3 19.12 2006

R avgrenset av $x^2+y^2+z^2=4$, $z=x^2+y^2$

paraboloide

$z=0$ og $z=1$.



 plan

c) Stjerneskov på

$\{x^2+y^2+z^2=4\}$-delen med tetthet $f(x,y,z)=z$.

Beregn mengden stjerneskov.

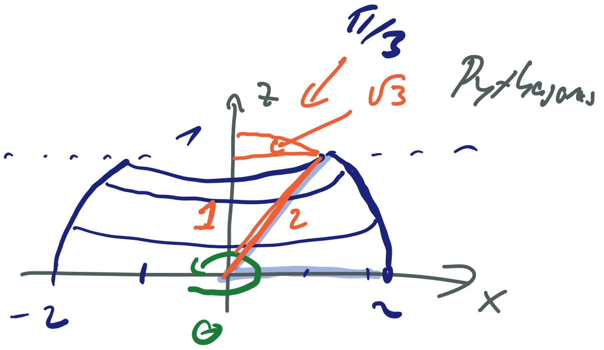
Løsn.

$\rho=2$

$$\boxed{2\cos(\phi)=1.}$$

$$r(\phi,\theta)=(2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$$

sf. koord.



$0\le\theta\le2\pi$

$\frac{\pi}{3}\le\phi\le\frac{\pi}{2}$

$$\left|\frac{\partial r}{\partial\phi}\times\frac{\partial r}{\partial\theta}\right|$$

$$=\begin{vmatrix} e_1 & e_2 & e_3 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$=|(4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi)|$$

$= 4\sin\phi.$

$$\iint_S f\, d\sigma = \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \underbrace{2\cos\phi}_{z}\; \underline{4\sin\phi\, d\phi\, d\theta}$$

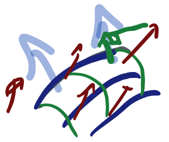
$$= 4\cdot 2\pi \sin^2\phi \Big|_{\pi/3}^{\pi/2} = 8\pi\left(1-\left(\frac{\sqrt{3}}{2}\right)^2\right) = 2\pi.$$

$$\frac{d}{d\phi}\sin^2\phi = 2\cos\phi\sin\phi$$

15.6 Fluks (flyt)

Fluksintegraler er et mål på flyten av et kont vektorfelt F gjennom en flate S:

$$\boxed{\iint_S F\cdot \underbrace{N\,d\sigma}_{\vec{dS}} = \iint_S F\cdot d\vec{S}}$$



der N er en enhetsnormal til S.

For en parametrisering $(s,t)\mapsto r(s,t)$:

$$\iint_{\mathcal{U}} \overrightarrow{F(r(s,t))} \cdot \frac{\pm\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}}{\cancel{\left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right|}}\; \underbrace{\cancel{\left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right|}\, ds\,dt}_{d\sigma}$$

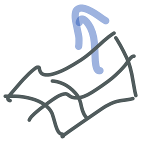
kontekstuelt!

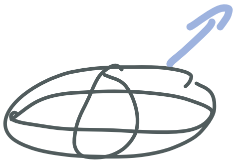
Def En flate er orienterbar dersom det finnes en kont (rettet) normal i hvert punkt på S.

I vårt tilfelle tilsvarer dette:

$\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}$ er kont og $\neq 0$ $\forall t,s$

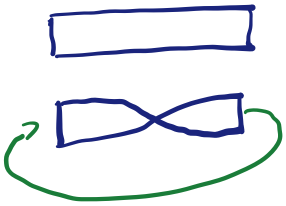
$+$ $(s,t) \mapsto r(s,t)$ injektiv.



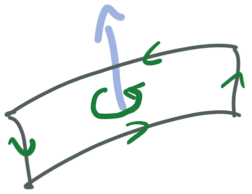


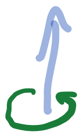
Men: finnes ikke-orienterbare flater

Møbius bånd



Hvordan vil retningen til N bli bevart?

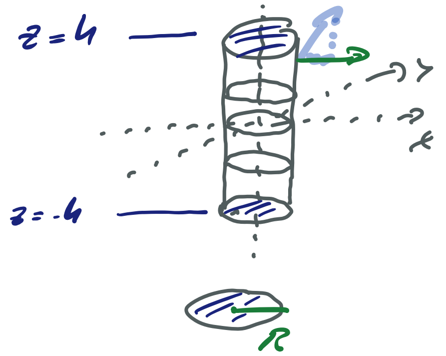




Eks Fluks av $F = id : \mathbb{R}^3 \to \mathbb{R}^3$, $(x,y,z) \mapsto (x,y,z)$ ut fra alle sider til sylinderen

$\{x^2+y^2 \le R^2,\ |z| \le h\}$

$S = S_{topp} \cup S_{side} \cup S_{bunn}$



Sylinderkoordinater

$S_{topp}: (a,\theta) \xmapsto{r} (a\cos\theta, a\sin\theta, h)$

$0 \le a \le R,\ 0 \le \theta < 2\pi$

$S_{bunn}: (a,\theta) \xmapsto{r} (a\cos\theta, a\sin\theta, -h)$

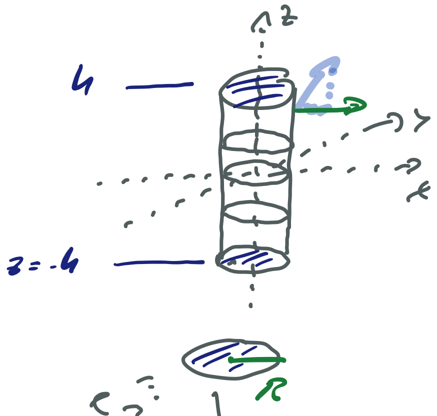
$0 \le a \le R,\ 0 \le \theta < 2\pi$

$S_{side}: (\theta, z) \longmapsto (R\cos\theta, R\sin\theta, z)$

$0 \le \theta < 2\pi,\ -h \le z \le h.$

Merk: S_{topp} og S_{bunn} flate

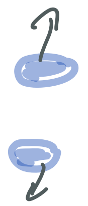
i (x,y)-planet



$\Longrightarrow \quad d\sigma = dA = dx\,dy = \underline{a\,da\,d\theta}$

$$\left|\frac{\partial r}{\partial a} \times \frac{\partial r}{\partial \theta}\right| = \left|\det \begin{pmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 0 \\ -a\sin\theta & a\cos\theta & 0 \end{pmatrix}\right|$$

$= |(0,0,a)| = a$

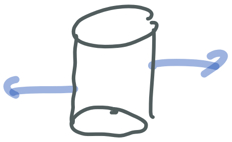


Enhetsnormal: $\pm(0,0,1)$

$S_{side}: \quad (\theta, z) \overset{r}{\mapsto} (R\cos\theta, R\sin\theta, z)$

$$\left|\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z}\right| = \left|\det\begin{pmatrix} e_1 & e_2 & e_3 \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|$$

$$= |(R\cos\theta, R\sin\theta, 0)| = R$$



<u>Enhetsnormal</u>: $(\cos\theta, \sin\theta, 0)$

$$\iint_{S_{topp}} F \cdot N \, d\sigma = \int_0^{2\pi}\int_0^R \underbrace{(a\cos\theta, a\sin\theta, h)}_{F(r(a,\theta))} \cdot \underbrace{(0, 0, a)\,da\,d\theta}_{\underbrace{(0,0,1)}_{N}\underbrace{a\,da\,d\theta}_{d\sigma}}$$

$$= \int_0^{2\pi}\int_0^R ha \, da\, d\theta = \underline{\pi h R^2}.$$

$$\iint_{S_{bunn}} F \cdot N \, d\sigma = \int_0^{2\pi}\int_0^R (-h)(-a)\, da\, d\theta = \underline{\pi h R^2}.$$

$$\iint_{S_{side}} F \cdot N \, d\sigma = \int_0^{2\pi}\int_{-h}^{h} \underbrace{(R\cos\theta, R\sin\theta, z)}_{F(r(\theta,z))} \cdot (R\cos\theta, R\sin\theta, 0)\, dz\, d\theta$$

$$= \int_0^{2\pi}\int_{-h}^{h} R^2 \, dz\, d\theta = \underline{4\pi h R^2}.$$

Så: $\iint_S F\cdot N\,dS = (4+1+1)\pi 4R^2 = \underline{6\pi 4R^2}.$

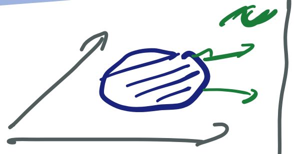
16.1 - 16.3 Divergens, curl og Green's teorem

F C^2 på $\bar{G}$

For et glatt vektorfelt F og en domene D med ∂D parametrisert ved en lukket kurve γ har vi sett på:

- Sirkulasjonen av F langs γ. $\quad \boxed{\oint_\gamma F\cdot T\,ds}$

- Fluksen av F ut fra D: $\quad \boxed{\oint_\gamma F\cdot N\,ds}$
  (tidligere gjennom en flate, nå ut fra en domene)



For $F = \nabla\phi$ fant vi $\oint_\gamma F\cdot T\,ds = 0$.

γ ← lukket

Finner en annen klasse F; $\oint_{v} \underline{F\cdot N}\,ds = 0$, og alle 'snille' F kan skrives $F = \underline{\nabla\phi} + \underline{\nabla\times\Psi}$

Hva er $\nabla\times\Psi$?

Def. For et vektorfelt $F \in C^1(\overline{U}, \mathbb{R}^n)$ er:

(i) $\nabla\cdot F \overset{def}{=} \sum_{j=1}^{n} \partial_{x_j} F_j = \partial_{x_1} F_1 + \dots + \partial_{x_n} F_n$

<u>divergensen</u> til F (obs skalar!)

(ii) • I $\underline{\underline{\mathbb{R}^3}}$ er $\nabla\times F \overset{def}{=} \det\begin{pmatrix} e_1 & e_2 & e_3 \\ \partial x_1 & \partial x_2 & \partial x_3 \\ F_1 & F_2 & F_3 \end{pmatrix}$

$$= \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3},\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1},\ \underline{\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}}\right)$$

<u>rotasjonen</u> (curl) til F (obs! vektor)

• I $\mathbb{R}^2$ er $\operatorname{curl}(F) \overset{def.}{=} \underline{\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}} = (\nabla\times F)\cdot e_3.$ ← $F=(F_1,F_2,0)$

<u>obs skalar!</u>

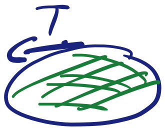
Merk: $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ konservativt i et enkelt sammenhengende område $\iff \frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}$.

$\implies$ F rotasjonsfritt $\iff$ F konservativt

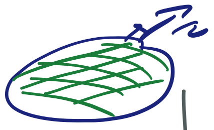
$\iff \underbrace{\oint_{r} F \cdot T\,ds = 0}_{\text{sirkulasjonsfritt.}}$

Dette er første variant av Greens teorem:

(I) Rotasjonen i det indre svarer til sirkulasjonen langs randen.



(II) Divergensen i det indre svarer til fluksen ut fra randen.



Teorem (Green) For et C^1-vektorfelt (P, Q): $U \subset \mathbb{R}^2 \to \mathbb{R}^2$ definert omkring $D \subset U$

med glatt rand ∂D parametrisert ved r, er:

$$\boxed{\oint_{\partial D} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}$$

Merk: $P\,dx + Q\,dy \overset{r=(x,y)}{=} (P, Q) \cdot \dot{r}\,dt$

(i) $F = (P, Q)$

$$\boxed{\oint_{\partial D} F \cdot \underline{T\,ds} = \iint_D \operatorname{curl}(F)\,dA}$$

(ii) $F = (Q, -P)$

$$\boxed{\oint_{\partial D} F \cdot \underline{N\,ds} = \iint_D \operatorname{div}(F)\,dA}$$

I 3D kalles

(i) Stokes' teorem

(ii) Gauß setning (divergensteorem)

Merk: $P\,dx + Q\,dy = -F_2\,dx + F_1\,dy$

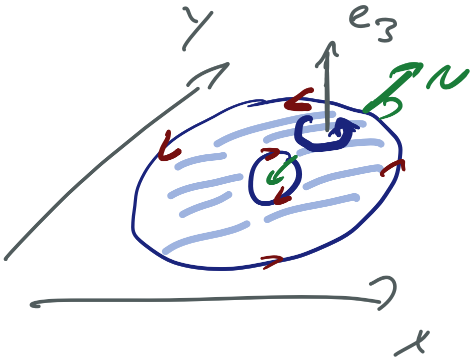
$= (F_1, F_2) \cdot (dy, -dx)$;

$$(dy, -dx) = \left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt$$

$$= \underbrace{\frac{(\dot{r}_2, -\dot{r}_1)}{|\dot{r}|}}_{N} \underbrace{|\dot{r}|\,dt}_{ds}$$

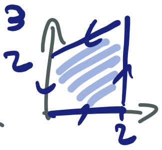
Hva er orienteringen til ∂D?

$$\underset{\text{def}}{N} = T \times e_3$$



Eks Beregn $\oint_C y^2\,dx + (2xy + x)\,dy = \cancel{(x}$

over



Løsn. Green: $\underbrace{\oint_{C} P\,dx + Q\,dy}_{\text{sirk.}} = \iint_{D} \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{rotasjon}} dx\,dy$

$P(x,y) = y^2, \quad Q(x,y) = 2xy + x$

$\frac{\partial Q}{\partial x} = 2y + 1, \quad \frac{\partial P}{\partial y} = 2y$

$\overset{\text{Green}}{\Longrightarrow} \quad (*) = \iint_{D} (2y + 1 - 2y)\,dA = \iint_{D} dA = S.$ ☒

Generelt: Ny måte å beregne areal på

dersom $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

F.eks: $(P,Q) = (0, x)$ eller $(P,Q) = (-y, 0)$

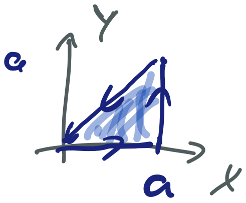
$\oint_{\partial D} x\,dy = A(D) \qquad -\oint_{\partial D} y\,dx = A(D)$

Eks. Oppg. 5 12/5 2003

a) Finn verdien til

$$\oint_C e^{y^2-x^2}\cos(2xy)\,dx + e^{y^2-x^2}\sin(2xy)\,dy$$

over C



Løsn. $F(x,y) = (\cos(2xy), \sin(2xy))e^{y^2-x^2}$

Green: $\oint_C F\cdot T\,ds = \iint_D \mathrm{curl}(F)\,dA = 0.$

$$\mathrm{curl}(F) \underset{\mathbb{R}^2}{=} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0, \text{ ettersom}$$

$$\frac{\partial F_2}{\partial x}(x,y) = 2y\cos(2xy)e^{y^2-x^2} - 2x\sin(2xy)e^{y^2-x^2},$$

$$\frac{\partial F_1}{\partial y}(x,y) = -2x\sin(2xy)e^{y^2-x^2} + 2y\cos(2xy)e^{y^2-x^2}.$$

Sirkulasjonen til F over C er 0.

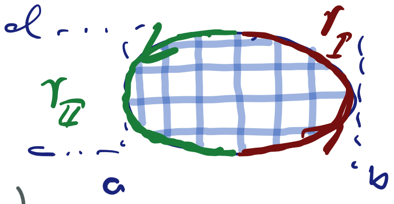
Bevis for Green's setning, for områder

$$D = \{(x,y): a \le x \le b,\ c(x) \le y \le d(x)\}.$$

c, d konst, <u>og</u> (samtidig)

$D = \{ (x,y) : c \le y \le d,\ a(y) \le x \le b(y) \}$

a, b kont.



$$\bullet \iint_D \frac{\partial Q}{\partial x}\, dA = \int_c^d \left(\int_{a(y)}^{b(y)} \frac{\partial Q(x,y)}{\partial x}\, dx \right) dy$$

$$= \int_c^d Q(x,y) \Big|_{x=a(y)}^{x=b(y)} dy$$

Fund. teor.

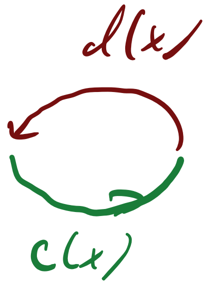
$$= \int_c^d \Big[Q(\underset{\gamma_I(y)}{b(y),y}) - Q(\underbrace{a(y),y}_{\gamma_{II}(y)}) \Big] dy$$

$$= \int_c^d Q(\gamma_I(y))\, dy \underset{\uparrow}{+} \int_c^d Q(\gamma_{II}(y))\, dy$$

$$= \int_\gamma Q\, dy.$$

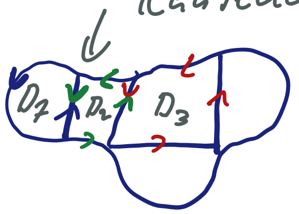
$$\bullet -\iint_D \frac{\partial P}{\partial y}\, dA = \overset{\text{helt likt}}{\cdots} =$$

$$= \int_\gamma P\, dx.$$



Så: $$\boxed{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_\gamma P\,dx + Q\,dy}$$ #

Merk: Flere områder kan betraktes sammen: kansellerer i kurveintegralene:



$$\oint_{\partial D_1} P\,dx + Q\,dy + \dots + \oint_{\partial D_n} P\,dx + Q\,dy$$

$$= \oint_{\partial D} P\,dx + Q\,dy.$$

Eks: kilder (<u>Newtonpotensialer</u>)

↑ koblet til fluks

$$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \quad \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$$

$$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

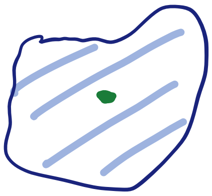
$$= \frac{1(\underline{x^2} + \underline{y^2}) - \underline{2x^2}}{(x^2+y^2)^2} + \frac{1(\underline{x^2} + \underline{y^2}) - \underline{2y^2}}{(x^2+y^2)^2}$$

$= 0$ overalt der F er definert.

Så flaks $\boxed{\oint_{\partial D} F \cdot N\, ds = \iint_{D} \operatorname{div} F\, dA = 0}$

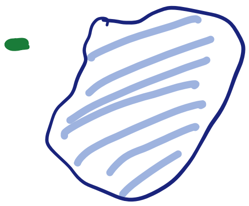
for alle enkelt lukkede kurver ∂D?

Nei, fordi: F ikke def. i origo!



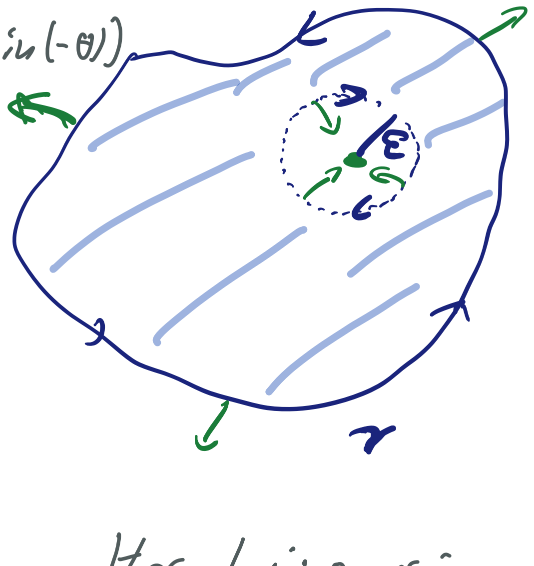
- Borte fra origo, $0 \notin D$, er dette OK

  ingen kilder i D (dvs $\operatorname{div} F = 0$ overalt)



- Kring origo, $0 \in D$, må vi utelukke (0,0):

$$\partial B_\varepsilon(0,0) : \theta \longmapsto \varepsilon(\cos(-\theta), \sin(-\theta))$$
$$0 \le \theta < 2\pi$$



Green: $$\oint_{\gamma} F \cdot N\, ds + \oint_{\partial B_\varepsilon} F \cdot N\, ds$$

$$= \iint_{D \setminus B_\varepsilon} \operatorname{div} F\, dA = 0$$

$$\Longrightarrow \oint_{\gamma} F \cdot N\, ds = - \oint_{\partial B_\varepsilon} F \cdot N\, ds$$

$\forall \gamma$ kring origo og $\forall \varepsilon > 0$ (likt)!

$$= + \int_0^{2\pi} \underbrace{\left(\frac{\varepsilon\cos(-\theta)}{\varepsilon^2}, \frac{\varepsilon\sin(-\theta)}{\varepsilon^2} \right)}_{F|_{\partial B_\varepsilon}} \cdot \underbrace{\left(\varepsilon\cos(-\theta), \varepsilon\sin(-\theta) \right)} d\theta$$

$$N\,ds = \frac{(-\varepsilon\cos(-\theta), \varepsilon\sin(-\theta))}{\underbrace{|\quad '' \quad - \quad|}_{\cancel{\varepsilon}}} \overset{\cancel{\varepsilon}}{|\dot{r}|}\, d\theta$$

$$= \int_0^{2\pi} d\theta = 2\pi$$ uavhengig av ε!

Så fluksen ut av enhver enkelt lukket kurve kring origo, er 2π. F har en kilde (source) i origo med styrke 2π.

- Generelt er $\mathrm{div}(F)$ et lokalt mål på produksjon i et punkt, mens fluks er et globalt mål.

- Generelt er curl(F) et lokalt mål på rotasjon, mens sirkulasjon er et globalt mål.

Egenskaper til curl og div ($\mathbb{R}^3$)

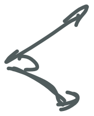
- $\overbrace{\operatorname{div}(\underbrace{\operatorname{curl}(\underbrace{F}_{\mathbb{R}^3})}_{\mathbb{R}^3})}^{\mathbb{R}} = 0 \;;\; \mathbb{R}$

- $\underbrace{\operatorname{curl}(\overbrace{\nabla \underbrace{\phi}_{\mathbb{R}}}^{\mathbb{R}^3})}_{\mathbb{R}^3} = 0 \;;\; \mathbb{R}^3$

Helmholtz:

$F: \mathbb{R}^3 \to \mathbb{R}^3$

$F = \underbrace{\nabla \phi}_{\text{rotasjonsfri}} + \underbrace{\nabla \times \psi}_{\text{divergensfri}}$



Eks. oppg. 4 20/8 2010

$F(x,y,z) = (x\cos(y^2),\; z - x^2 y \sin(y^2),\; y)$

(a) Finn curl(F).

(b) Beregn $\int_\gamma F \cdot T\, ds$ for

$\gamma : t \longmapsto (\sin(t),\; \sin(2t),\; t(\pi - 2t))$

$t \in (0, \frac{\pi}{2})$

Løsn. (a) $\mathrm{curl}(F) = \nabla \times F$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= (1 - 1, 0 - 0, -2xy\sin(y^2) + 2xy\sin(y^2))$$

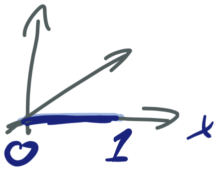
$$= (0, 0, 0)$$ $\underline{F \text{ rotasjonsfri}}$

(b) $\mathrm{curl}(F) = 0 \iff$ F konservativ

$\mathbb{R}^3$ enkeltsam.

$\iff \int_\gamma F \cdot T\,ds$ veiuavhengig

$\gamma(0) = (0,0,0)$, $\gamma(\frac{\pi}{2}) = (1, 0, 0)$



Velg $\tilde{\gamma}(x) = (x, 0, 0)$, $0 \le x \le 1$.

$$\boxed{F(x,y,z) = (\underline{x\cos(y^2)}, \underline{z - x^2 y\sin(y^2)}, \underline{y})}$$

$$\int_\gamma F \cdot T\,ds \underset{\substack{\uparrow \\ F \text{ kons}}}{=} \int_{\tilde{\gamma}} F \cdot T\,ds = \int_0^1 F(\tilde{\gamma}(x)) \cdot \tilde{\gamma}'(x)\,dx$$

$$= \int_0^1 (x, 0, 0) \cdot (1, 0, 0)\,dx = \frac{1}{2}.$$

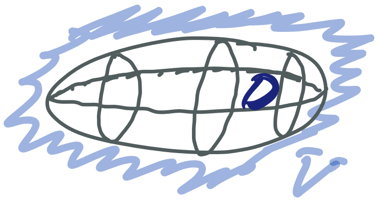
Alt. $\phi(x,y,z) = yz + \frac{x^2}{2}\cos(y^2)$ potensial,

dvs $\nabla \phi = F$.

$$\Rightarrow \int_r F\cdot T\,ds = \phi(1,0,0) - \phi(0,0,0)$$
$$= \frac{1}{2}. \qquad \#$$

16.4 Divergensteoremet (Gauß' setning)

$D \subset \mathbb{R}^3$ kompakt,
lukket, begrenset



med orienterbar rand ∂D

gitt ved en glatt flate $\left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| \neq 0.$

$F \in C^1(V, \mathbb{R}^3)$, V åpen med $D \subset V$.

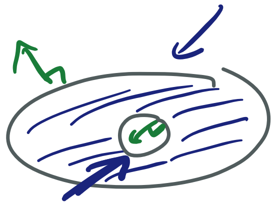
Da gjelder:

$$\boxed{\iint_{\partial D} F\cdot N\,d\sigma = \iiint_D \operatorname{div}(F)\,dV}$$

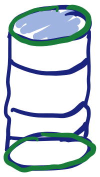
'Fluksen til F ut av ∂D'
(N peker ut av D)

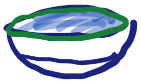
(i) $N = \pm \dfrac{\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}}{\left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right|}$, $d\sigma = \left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| dt\,ds$

(ii) ∂D kan bestå av flere deler



(iii) ∂D kan være stykkevis glatt:





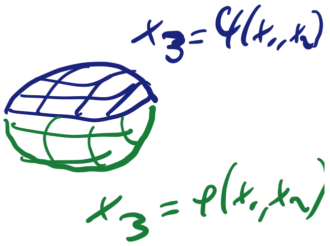
Bevis lik Green's, for domener av typen $D = \{(x_1, x_2, x_3) : \varphi_k(x_i, x_j) \leq x_k \leq \psi_k(x_i, x_j)\}$

$(x_i, x_j) \in A$, $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k \neq i$.

Skriv $F = \underbrace{(F_1, 0, 0)}_{G_1} + \underbrace{(0, F_2, 0)}_{G_2} + \underbrace{(0, 0, F_3)}_{G_3}$

og vis at
$$\boxed{\iint_{\partial D} G_j \cdot N \, d\sigma = \iiint_D \frac{\partial F_j}{\partial x_j} \, dV}$$

$$\xRightarrow{\sum_{j=1}^{3}} \iint_{\partial D} F \cdot N \, d\sigma = \iiint_D div(F) \, dV.$$

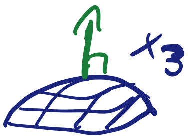


Viser for $j = 3$:

...

$$\iiint_D \frac{\partial F_3}{\partial x_3}\, dV = \iint_A \left(\int_{\rho(x_1,x_2)}^{\psi(x_1,x_2)} \frac{\partial F_3}{\partial x_3}\, dx_3 \right) dx_1\, dx_2$$

$$\underset{\substack{\uparrow \\ \text{fund.setn}}}{=} \iint_A \Big(\underline{F_3(x_1,x_2,\psi(x_1,x_2))} - F_3(x_1,x_2,\rho(x_1,x_2)) \Big) \underset{\nearrow}{} dx_1\, dx_2$$



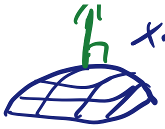
'Toppen' $\{ \underbrace{(x_1,x_2,\psi(x_1,x_2))}_{r(x_1,x_2) \text{ på grafform}} : (x_1,x_2)\in A\}$

med tangentene $\frac{\partial r}{\partial x_1} = \left(1,0,\frac{\partial \psi}{\partial x_1}\right)$, $\frac{\partial r}{\partial x_2} = \left(0,1,\frac{\partial \psi}{\partial x_2}\right)$

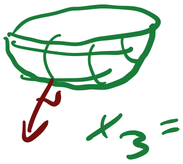
og $\frac{\partial r}{\partial x_1} \times \frac{\partial r}{\partial x_2} = \left(-\frac{\partial \psi}{\partial x_1}, -\frac{\partial \psi}{\partial x_2}, \underset{\substack{\uparrow \\ \text{peker opp}}}{1}\right)$

$$\iint_A F_3(x_1,x_2,\psi(x_1,x_2))\, dx_1\, dx_2$$

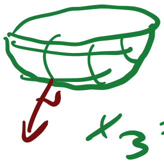
$$= \iint_A (0,0,F_3)\Big|_{x_3=\psi(x_1,x_2)} \cdot \left(-\frac{\partial \psi}{\partial x_1}, -\frac{\partial \psi}{\partial x_2}, 1\right) dx_1\, dx_2$$

$= \iint G_3 \cdot N \, d\sigma$ på  $x_3 = \psi(x_1, x_2)$

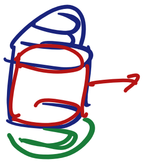
på 'bunnen', likt:

 $x_3 = \varphi(x_1, x_2)$ $\quad -\iint_A F_3(x_1, x_2, \varphi(x_1, x_2)) \, dx_1 \, dx_2$

$$\underset{F_3 \cdot (-1)}{=} \iint_A (0, 0, F_3) \cdot \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, -1 \right)$$

$= \iint G_3 \cdot N \, d\sigma$ på  $x_3 = \varphi(x_1, x_2)$

- Hva hvis det finnes en vertikal del av ∂D?



$N = (N_1, N_2, 0)$

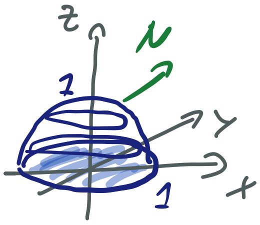
$\perp \; (0, 0, F_3)$.

Så $\iiint_D \frac{\partial F_3}{\partial x_3} \, dV = \iint_{\partial D} G_3 \cdot N \, d\sigma$

$$\sum_{j=1}^{3} \Longrightarrow \boxed{\iiint_D \operatorname{div}(F) \, dV = \iint_{\partial D} F \cdot N \, d\sigma}$$

Eks. Oppg. 6 8/8 2002

$F(x,y,z) = (x^3 z, y^3 z, x^2+y^2)$



$D = \{x^2+y^2+z^2 \le 1, z \ge 0\}$

Beregn fluksen ut av toppen til D:

$T = \{x^2+y^2+z^2 = 1, z \ge 0\}$.

Løsn. Bunn: $B = \{x^2+y^2 \le 1, z = 0\}$

$\partial D = T \cup B$ stykkevis glatt flate, $F \in C^1(\mathbb{R}^3, \mathbb{R}^3$

Kan bruke div. setn.: $\boxed{\iint_{\partial D} F\cdot N\,d\sigma = \iiint_D \mathrm{div}(F)\,dV}$

$\Rightarrow$ Fluks (ut av T): $\iint_T F\cdot N\,d\sigma = \underbrace{\iiint_D \mathrm{div}(F)\,dV}_{I} - \underbrace{\iint_B F\cdot N\,d\sigma}_{II}$

① $\mathrm{div}(F) = 3x^2 z + 3y^2 z + 0 = 3z(x^2+y^2)$.

I: $\iiint_D \mathrm{div}(F)\,dV = 3\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \overbrace{\rho\cos\phi}^{z}\ \overbrace{\rho^2\sin^2\phi}^{x^2+y^2}\ \overbrace{\rho^2\sin\phi}^{dV}$ $d\rho\, d\theta\, d\phi$

$x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\sin\phi$

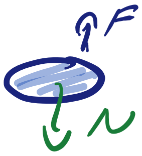
$$= 6\pi \left(\underbrace{\int_0^{\pi/2} \cos\phi \sin^3\phi \, d\phi}_{\frac{\sin^4\phi}{4} \rightsquigarrow \frac{1}{4}} \right) \left(\underbrace{\int_0^1 \rho^5 d\rho}_{\frac{1}{6}} \right)$$

$z = \rho \cos\phi$

$$= 6\pi \cdot \frac{1}{4} \cdot \frac{1}{6} = \frac{\pi}{4}.$$

- - - - - - - - -

② $$\iint_B F \cdot N \, d\sigma = \iint_B F \cdot (0,0,-1) \, dA$$



$$= \iint_{\substack{0 \le x^2+y^2 \le 1 \\ (z=0)}} F \cdot (0,0,-1) \, dA \overset{\uparrow}{=} -\int_0^{2\pi} \int_0^1 r^2 \underbrace{r \, dr \, d\theta}_{dA}$$

pol. koord.

$$= -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}.$$

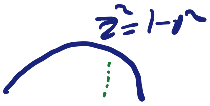
Svar: $\iint_T F \cdot N \, d\sigma = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}.$

Alt. til ① med sylinderkoord.

$x = r\cos\theta$ | $0 \le \theta \le 2\pi$

$y = r\sin\theta$ | $0 \le r \le 1$

$z = z$ | $0 \le z \le \sqrt{1-r^2}$



$dV = \boxed{r} \, dr \, d\theta \, dz$

$$\iiint_D \underbrace{\mathrm{div}(F)}_{3z(x^2+y^2)}\, dV = 3\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} z\, r^2\, r \; dz\, dr\, d\theta$$

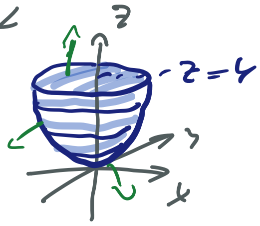
$$= 6\pi \int_0^1 r^3 \left[\frac{z^2}{2}\right]_0^{\sqrt{1-r^2}} dr = 3\pi \int_0^1 r^3(1-r^2)\, dr$$

$$= 3\pi\left[\frac{1}{4} - \frac{1}{6}\right] = \frac{\pi}{4}.$$

#

Eks Fluks ut av (over) flaten til legemet

$$D = \{(x,y,z) : \overbrace{x^2+y^2}^{r^2} \overset{\le z \le 4}{\le z \le 4}\}$$



for vektorfelt $F : (x,y,z) \mapsto (x,y,3)$

Divergensteoremet / Gauß' setning

$$\iint_{\partial D} F \cdot N\, d\sigma = \iiint_D \underbrace{\mathrm{div}(F)}_{\sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}}\, dV$$

$$= 1 + 1 + 0 = 2$$

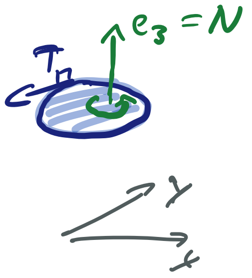
$$= 2\int_0^{2\pi}\int_0^2\left(\int_{r^2}^4 dz\right) r\, dr\, d\theta = 4\pi\int_0^2 (4-r^2)\, r\, dr$$

syl. koord.

$$= 4\pi\left[2r^2 - \frac{r^4}{4}\right]_0^2 = 4\pi[8-4] = 16\pi.$$

#

16.5 Kelvin-Stokes' setning



Husk Green i sirkulasjonsform:

$$\int_{\partial S} F\cdot T\,ds = \iint_S \underline{\text{curl}(F)}\,d\sigma$$

$$\text{curl}(F_1, F_2, 0)\cdot e_3 = \underline{\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}}$$

Nå, trekk flaten S opp til 3D

Stokes' er Green's for $F = (F_1, F_2, F_3)$,

og S stykkevis glatt i 3D.

Teorem (Stokes')

(i) $r : A \subset \mathbb{R}^2 \to S = r(A) \subset \mathbb{R}^3$

$(t,s) \mapsto r(t,s)$ glatt flate

(kont. der. bar med $\left|\frac{\partial r}{\partial t}\times\frac{\partial r}{\partial s}\right| \neq 0$)

og injektiv.

orientert ved $N = \dfrac{\frac{\partial r}{\partial t}\times\frac{\partial r}{\partial s}}{|\text{ -"- }|}$

med (orientert) rand $\partial S = r(\partial A)$.

(ii) $F \in C^2(\bar{V}, \mathbb{R}^3)$, $S \subset \bar{V}$ åpen.

Da er:

$$\boxed{\oint_{\partial S} F \cdot T \, ds = \iint_S \operatorname{curl}(F) \cdot N \, dS}$$

Bevis (for F og S C^2)

Skriv $S = \bigcup_{j=1}^{N} S_j$, der hver S_j er på grafform:

$x = f(y,z)$, $y = f(x,z)$, eller $z = f(x,y)$.

Kan alltid gjøres da $\left|\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s}\right| \neq 0$,

ettersom

$$\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{pmatrix}$$

$$= \left(\boxed{\frac{\partial y}{\partial t}\frac{\partial z}{\partial s} - \frac{\partial z}{\partial t}\frac{\partial y}{\partial s}}, \boxed{\frac{\partial z}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial x}{\partial t}\frac{\partial z}{\partial s}}, \boxed{\frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial z}{\partial t}\frac{\partial x}{\partial s}} \right.$$

$$\uparrow \qquad \nearrow \qquad \nearrow$$

Minst én av disse skilt fra null i hvert punkt

Jacobideterminant til $\Phi : (t,s) \mapsto (x,y)$

$$D\phi = \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{pmatrix}.$$

Inverse f. teorem $\Longrightarrow$

$(t,s) \mapsto (x,y)$

C^2-bijeksjon

Nå kan skifte (t,s) mot (x,y):

$$r(t,s) = r(t(x,y), s(x,y)) = \tilde{r}(x,y) = (x,y,f(x,y))$$

(eller tilsv. for (x,z) alt. (y,z)).

Så anta $S_j = \{z = f(x,y) : (x,y) \in A\}$

sånn at ∂S_j er gitt ved

$$r(\tau) = (x(\tau), y(\tau), f(x(\tau), y(\tau)))$$

Så: $$\oint_\gamma F \cdot T\,ds = \oint_\gamma F_1 \underline{\dot{x}} + F_2 \underline{\dot{y}} + F_3 (f'_x \underline{\dot{x}} + f'_y \underline{\dot{y}})\,d\tau$$

$$= \oint_\gamma \underbrace{(F_1 + F_3 f'_x)}_{P} \underbrace{\dot{x}\,d\tau}_{dx} + \oint_\gamma \underbrace{(F_2 + F_3 f'_y)}_{Q} \underbrace{\dot{y}\,d\tau}_{dy}$$

Green

=

i 2D!

(x,y)

$$\iint_{A} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = (*)$$

(x,y)

$F_j(x,y,f(x,y))$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(F_2 + F_3 f'_y \right) = \underline{\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} f'_x}$$

$$+ \underline{\frac{\partial F_3}{\partial x} f'_y} + \cancel{\frac{\partial F_3}{\partial z} f'_x f'_y} + \cancel{F_3 f''_{yx}}$$

$$-\frac{\partial P}{\partial y} = -\frac{\partial}{\partial y}\left(F_1 + F_3 f'_x \right) = \underline{-\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z} f'_y}$$

$$\underline{-\frac{\partial F_3}{\partial y} f'_x} - \cancel{\frac{\partial F_3}{\partial z} f'_y f'_x} - \cancel{F_3 f''_{xy}}$$

Nå: normalen N til en graf $z=f(x,y)$

er $\dfrac{(-f'_x, -f'_y, 1)}{|\text{—}''\text{—}|}$

$$\underline{N\,d\sigma = (-f'_x, -f'_y, 1)\,dx\,dy}$$

Grønne termer : $\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right) f'_x + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) f'_y$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \cdot 1$$

$$= \text{curl}(F) \cdot (-f'_x, -f'_y, 1) \quad !$$

Sei $$\boxed{\oint_\gamma F \cdot T\,ds = \iint_A \text{curl}(F) \cdot N\,d\sigma}$$ ☒

Eks. Beregn $\oint_{\partial S} F \cdot T\,ds$ når $F(x,y,z) = (3z, 5x, -2y)$

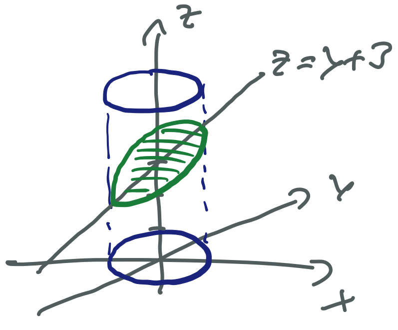
og ∂S er skjæringen mellom flatene

$x^2 + y^2 = 1$ og planet $z = y + 3$.

Anta at ∂S er orientert motvols ovenifra.

Løsn. $F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$

$$\text{curl}(F) = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ 3z & 5x & -2y \end{pmatrix}$$



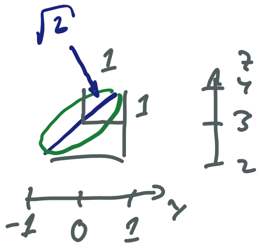
$= (-2, 3, 5)$ konstant.

Bruker Stokes på 

planet $0 - y + z = 3$ har $N = \frac{(0,-1,1)}{\sqrt{2}}$

Stokes $\Longrightarrow$ $$\oint_{\partial S} F \cdot T \, ds = \iint_S \operatorname{curl}(F) \cdot N \, d\sigma$$

$$= \iint_S (-2,3,5) \cdot \frac{(0,-1,1)}{\sqrt{2}} \, d\sigma = \sqrt{2} \underbrace{\iint_S d\sigma}_{\text{areal til}} = (*)$$

Alt 1 

Ellipse med halvakser $\sqrt{2}$ (i '$z = 3 + y$'-retning) og 1 (i x-retning).

$$\underline{A(S) = 1 \cdot \sqrt{2} \cdot \pi} \Longrightarrow \underline{(*) = 2\pi.}$$

Alt 2: Parameteriser S:

$$r(r, \theta) = (r\cos\theta, r\sin\theta, \overbrace{r\sin\theta + 3}^{z = y + 3})$$

$0 \le r < 1$, $0 \le \theta < 2\pi$.

$$\left| \frac{\partial r}{\partial r} \times \frac{\partial r}{\partial \theta} \right| = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & \sin\theta \\ & & \end{vmatrix}$$

$$\left| -r\sin\theta \quad r\cos\theta \quad r\cos\theta \right|$$

$$= |(0, -r, r)| = \sqrt{2}\, r.$$

$$\Rightarrow \boxed{d\sigma = \sqrt{2}\, r\, dr\, d\theta}$$

Areacalc: $A(S) = \int_0^{2\pi} \int_0^{1} \sqrt{2}\, r\, dr\, d\theta = 2\pi \sqrt{2} \frac{1}{2}$

$$= \sqrt{2}\, \pi.$$

$$\Rightarrow (*) = 2\pi.$$
