'Minste Taylor' (Middelverdisetn.)

F deriverbar $\bar{U} \subset \mathbb{R}^n \to \mathbb{R}$, $x, y \in \bar{U}$,
med $\gamma(t) = (1-t)x + ty \in \bar{U}$, $t \in [0,1]$.

Da $\exists\, \tau \in (0,1)$ og $c = \gamma(\tau)$:

$$\boxed{F(y) - F(x) = \nabla F(c) \cdot (y-x)}$$

vektorer!

Merk: Med $x = x_0$, $y = x_0 + h$, får vi:

$$\boxed{F(x_0+h) = F(x_0) + \nabla F(c) \cdot h}$$

Bevis: La $f : F \circ \gamma : [0,1] \to \mathbb{R}$

deriverbar med $f'(t) = \nabla F(\gamma(t)) \cdot \dot{\gamma}(t)$
$= \nabla F(\gamma(t)) \cdot (y-x)$.

$$\boxed{\gamma(t) = (1-t)x + ty}$$

Middelv. setn. $\mathbb{R} \to \mathbb{R}$:

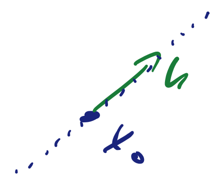
$\exists\, \tau \in (0,1)$: $\underbrace{f(1)}_{F(y)} - \underbrace{f(0)}_{F(x)} = f'(\tau)(1-0) = \nabla F(\overbrace{\gamma(\tau)}^{c}) \cdot (y-x)$.

$\Rightarrow \quad F(y) - F(x) = DF(c)\cdot(y-x).$

---

obs! Finnes ikke for $F:\mathbb{R}^n \to \mathbb{R}^m$.
Kan ikke finne en felles τ for alle F_j.

Nå: la $r(t) = \vec{x}_0 + t\vec{h}$ og bruk samme idé som ovenfor.

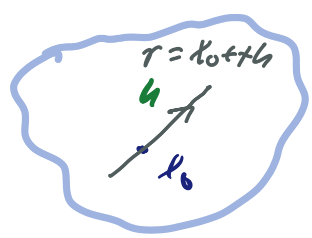


$\rightsquigarrow$ Taylor for $F: U \subset \mathbb{R}^n \to \mathbb{R}$.

Def. $F \in C^k(U,\mathbb{R}) \overset{\text{def.}}{\iff} \underbrace{\partial x_i \dots \partial x_j}_{k \text{ stykk}} F$ kont.

Eks. $F \in C^2(\mathbb{R}^2,\mathbb{R})$ dersom $F''_{xx}, F''_{xy}, F''_{yx}, F''_{yy}$ er kont.

Kj.r. $\begin{cases} F \in C^k \\ r \in C^\infty \end{cases} \Rightarrow f = F \circ r \quad C^k(\mathbb{R},\mathbb{R})$



Taylor $\mathbb{R}\to\mathbb{R}$:

$$f(t) = f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \dots + \frac{t^k}{k!} f^{(k)}(\tau)$$

τ mellom 0 og t.

Men $f(0) = F(r(0)) = F(x_0)$

$$f'(0) = F(r)'(0) = \nabla F(r(0)) \cdot \dot{r}(0) = \nabla F(x_0) \cdot h$$

Så for $t = 1$ blir begynnelsen på Taylorrekken:

$$\boxed{\underbrace{F(x_0 + h)}_{f(1)} = \underbrace{F(x_0) + h \cdot \nabla F(x_0)}_{\text{Taylorpolynom av grad } 1} + \dots}$$

Kan fortsette så langt F er kont. deriverbar.

Men hva er $f''(0)$?

Se på $F(x_1, x_2)$, $h = (h_1, h_2)$, $r(t) = (x_1 + th_1, x_2 + th_2)$

$$f' = \nabla F(r) \cdot \dot{r} = [(\partial_{x_1} F)(r), (\partial_{x_2} F)(r)] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \boxed{h_1 \partial_{x_1} F(r) + h_2 \partial_{x_2} F(r).}$$

$$f'' = h_1 \nabla(\partial_{x_1} F)(r) \cdot \dot{r} + h_2 \nabla(\partial_{x_2} F)(r) \cdot \dot{r}$$

$$= h_1^2 \partial_{x_1}^2 F(r) + \underbrace{h_2 h_1 \partial_{x_2} \partial_{x_1} F(r) + h_1 h_2 \partial_{x_1} \partial_{x_2} F(r)}_{2 h_1 h_2 \partial_{x_1} \partial_{x_2} F(r)}$$

$+ h_n^2 \partial_{x_n}^2 F(x)$

Med $D^2 F = \begin{bmatrix} \partial_{x_1}^2 F & \partial_{x_1}\partial_{x_n} F \\ \partial_{x_n}\partial_{x_1} F & \partial_{x_n}^2 F \end{bmatrix}$ får

'Hessian'

$$F(x_0+h) = F(x_0) + h \cdot DF(x_0) + [h] \begin{bmatrix} D^2F \end{bmatrix} \begin{bmatrix} h \end{bmatrix} + O(|h|^3)$$

dersom $F \in C^3$

der $|O(|h|^3)| \leq C|h|^3$

obs! for små h

C bestemmes av 3:e ordens derivate til F nært x_0.

Taylorpolynomer er unike (har vi funnet et, er det det riktige.

Eks. Taylor av $\cos(x)\sin(y)$ i $(0,0)$.

Vet $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \frac{y^7}{7!} + \dots$

$$\Rightarrow \cos(x)\sin(y) = \underbrace{y}_{O(|(x,y)|)} - \underbrace{\frac{x^2y}{2} - \frac{y^3}{3!}}_{O(|(x,y)|^3)} + O(|(x,y)|^5)$$

<u>Obs!</u> Polynomer er sine egne Taylorutv. kring origo.

$$D^2F(x,y) = \begin{bmatrix} 0 & 2y \\ 2y & 2x \end{bmatrix}$$

<u>Eks.</u> $F(x,y) = x + xy^2$, $DF(x,y) = (1+y^2, 2xy)$

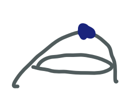
$$F(0+h_1, 0+h_2) = F(h_1,h_2) = \underbrace{h_1}_{DF(0,0)\cdot(h_1,h_2)} + \underbrace{h_1h_2^2}_{\text{Fra } D^3F\ldots}$$

<u>Men</u> $F(1+h_1, 1+h_2) =$

$$F(1,1) + (h_1,h_2)\cdot(2,2) \cdot O(|(h_1,h_2)|^2)$$

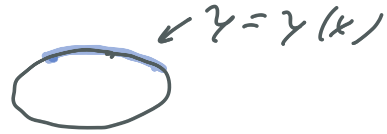
$$= \underbrace{2 + 2h_1 + 2h_2}_{\text{TP til } F \text{ gjennom } (1,1,2)} + O(h_1^2 + h_2^2)$$

Grunnleggende egenskaper til kont. funksj.



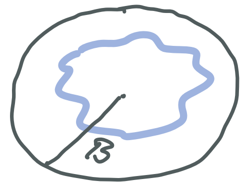
– Ekstremalverdisetn.

- Omvendte funksjonsteorem. 

- Implisitte —"— 

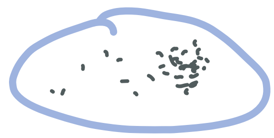
Def. $U \subset \mathbb{R}^n$ <u>begrenset</u> $\overset{\text{def.}}{\iff}$

$\exists B > 0$; $|x| \le B \quad \forall x \in \bar{U}$



Bolzano – Weierstrass

$\bar{U} \subset \mathbb{R}^n$ begrenset $\Rightarrow$ hver følge $(x_j)_j \subset \bar{U}$ har en konvergent delfølge.



Bevis B-W sann i én dimensjon. (MA1102)

Skriv $\bar{x}_j = (x_j^1, x_j^2, \dots, x_j^n) \in \mathbb{R}^n$.

$\mathbb{R}$

$(x_j^1)_j$ begr. følge i $\mathbb{R}$ $\Rightarrow$ har konv. delfølge i $\mathbb{R}$.

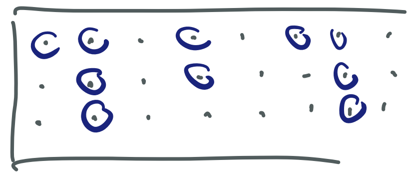
Betrakt x_j^2 for samme delfølge

$\overset{\text{B-W 1 dim}}{\Longrightarrow}$ har en konv. delfølge.

$\overset{\text{o.i.v.}}{\Longrightarrow}$ $\exists$ delfølge til $(x_j^1, x_j^2, \dots, x_j^n)$

$(x^1, x^2, \dots, x^n) = \vec{x}$

konv. i hver komponent.



$\Longrightarrow \exists\, x \in \bar{U}$; $x_j \longrightarrow x$.

Heine-Borel U lukket og begrenset i $\mathbb{R}^n$

$\Longleftrightarrow$ U er kompakt,

dvs. hver følge $(x_j)_j \subset \bar{U}$ har en konv.

delfølge : $x_{j_k} \xrightarrow{k \to \infty} x \in \bar{U}$.

Ekstremalverdisetningen.

$F : \bar{U} \subset \mathbb{R}^n \to \mathbb{R}$ med U kompakt

(lukket og begrenset)

F kont på $\bar{U}$ $\Longrightarrow \exists\, x_0 \in \bar{U}$: $F(x_0) = \max_{x \in \bar{U}} F(x)$.

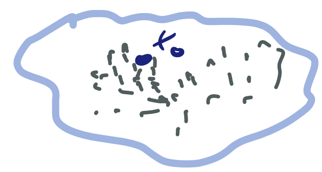
Bevis $\exists \sup_{x \in \bar{U}} F(x)$, velg $(x_j)_j$ sånn at

$$\boxed{\lim_{j \to \infty} F(x_j) = \sup_{x \in \bar{U}} F(x)}$$

$(x_j)_j \subset \bar U$ kompakt $\xRightarrow[HB]{BW}$ $\exists$ konv

delfølge $(x_{j_k})_k$; $x_{j_k} \to x_0 \in \overline{\bar U} = \bar U$

U lukket



F kont. $\boxed{F(x_{j_k}) \to \underbrace{F(x_0)}_{\mathbb{R}}}$

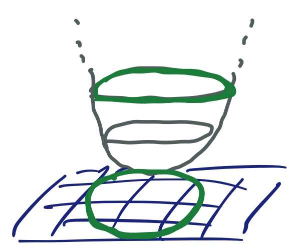
da $x_{j_k} \to x_0$.

$\forall x \in \bar U$ gjelder $F(x) \le \sup_{x \in \bar U} F(x) = \lim_{j\to\infty} F(x_j)$

$$= \lim_{k\to\infty} F(x_{j_k}) = F(x_0).$$

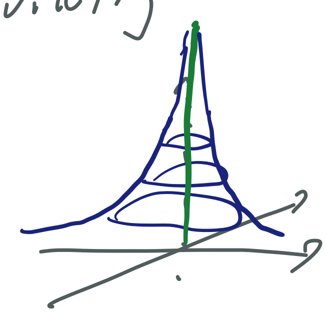
Så $F(x_0) = \max_{x \in \bar U} F(x)$. #

Tenk på: $F(x,y) = x^2 + y^2$



kompakt viktig

$\frac{1}{|(x,y)|}$



Nå: IFT (Inverse/Implicit Function Theorem)

Disse er ekvivalente.

Def. $\boxed{id : x \mapsto x. \quad \mathbb{R}^n \to \mathbb{R}^n}$

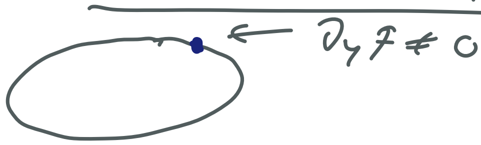
Inverse f.sjon, idé: $f : \mathbb{R} \to \mathbb{R}$, $f'(x_0) \neq 0$

$\implies \exists f^{-1}$ nært $y_0 = f(x_0)$, og

$\frac{d}{dx}$ på $f^{-1} \circ f = id$

$$\implies \left((f^{-1})' \circ f\right) \cdot f' = 1 \implies \boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}$$

Implisitte, idé: $\boxed{F(x,y) = 0, \ \partial_y F(x_0, y_0) \neq 0}$



$F : \mathbb{R}^2 \to \mathbb{R}$

$\partial_y F$ kont $\implies \partial_y F(x,y) \neq 0$ nært (x_0, y_0).

Betrakt x som en parameter.

Omvendte f.sjon. i én variabel

$\Longrightarrow \forall x$ nært x_0 $\exists\, \Phi_x : F \longrightarrow y,$

dvs $y = \Phi_x(F)$ der $F(x,y) = 0 \Longrightarrow \boxed{y = \Phi_x(0)}$

Så y bestemt av x nært x_0.

$F(x,y) = 0 \iff \underline{F(x, y(x)) = 0}$ nært (x_0, y_0)

$$\frac{d}{dx} \overset{\text{kj.r.}}{\rightsquigarrow} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y'(x) = 0$$

$$\Longrightarrow y'(x) = -\frac{\partial F/\partial x}{\partial F/\partial y}$$; (x,y) på kurven.

Inverse funksjonssetningen for $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$

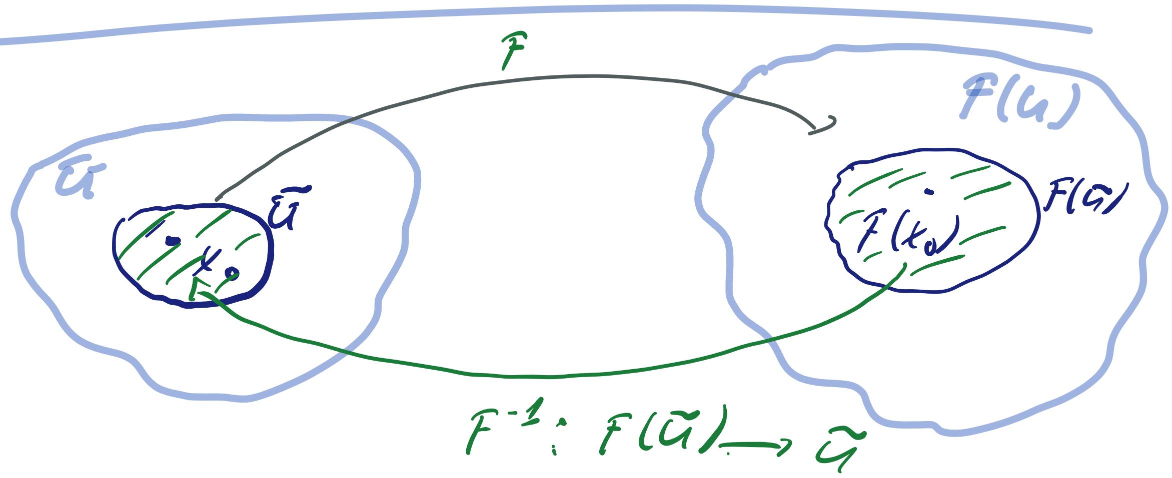
$U \subset \mathbb{R}^n$ åpen, $x_0 \in \bar{U}$ og $F \in C^1(\bar{U}, \mathbb{R}^n)$

med $\underbrace{DF[x_0]}_{n \times n\text{-Jakobi-matrise}}$ invertibar $\mathbb{R}^n \to \mathbb{R}^n$,

$\Longrightarrow \exists\, \tilde{U} \ni x_0 : F : \tilde{U} \longrightarrow F(\tilde{U})$ er

invertibar, og

$$\boxed{D(F^{-1}) = (DF)^{-1} \circ F^{-1} \text{ kont på } F(\tilde{U}).}$$



$$\boxed{DF(x_0): \mathbb{R}^n \longleftrightarrow \mathbb{R}^n}$$

Implisitte f. setn. for $F: U \subset \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$

$U \subset \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ åpen, $u_0 = (\underset{\mathbb{R}^n}{\underset{\cap}{x_0}}, \underset{\mathbb{R}^m}{\underset{\cap}{y_0}}) \in U$,

$F \in C^1(U, \mathbb{R}^m)$ med $F(x_0, y_0) = \underset{\mathbb{R}^m}{0}$ og

$D_y F(x_0, y_0)$ omvendbar $\mathbb{R}^m \to \mathbb{R}^m$.

$\Longrightarrow \exists\ \tilde{U} = \tilde{B}_{x_0} \times \tilde{B}_{y_0} \subset \mathbb{R}^n \times \mathbb{R}^m$ og

$\Phi \in C^1(\tilde{B}_{x_0}, \tilde{B}_{y_0})$: $F(x, \Phi(x)) = 0$

inneholder alle løsninger til $F(x,y)=0$ i $\bar{U}_1$

og $D\Phi = -[D_y F(\Phi)]^{-1} D_x F(\Phi)$.

Bevis (gitt omvendte funksjon.)

$$\text{La } G: \underset{\mathbb{R}^n \times \mathbb{R}^m}{(x,y)} \longrightarrow \underset{\mathbb{R}^n \times \mathbb{R}^m}{(\overset{G_1}{x}, \overset{G_2}{F(x,y)})}$$

$$\Longrightarrow G \in C^1(\bar{U}, \mathbb{R}^{n+m}) \text{ med}$$

$$\begin{bmatrix} 1 & \cdots & 0 \\ 0 & & 1 \end{bmatrix}$$

$$DG = \begin{bmatrix} D_x G_1 & D_y G_1 \\ D_x G_2 & D_y G_2 \end{bmatrix} = \begin{bmatrix} [Id] & [0] \\ [D_x F] & [D_y F] \end{bmatrix}$$

$$\Longrightarrow \det(DG) = \det(D_y F) \neq 0 \quad i \ (x_0, y_0).$$

antatt inv.bar

omv. setn.

$\Longrightarrow \quad \exists \ C^1$-invers

$$G^{-1}: \underset{(x,\, F(x,y))}{(v, w)} \longrightarrow (\underset{x}{G_1^{-1}(v,w)}, \underset{y}{G_2^{-1}(v,w)})$$

$$y = G_2^{-1}(x, F(x,y)) = G_2^{-1}(x, 0) \text{ der } F(x,y) = 0.$$

Så $y = \Phi(x) \overset{\text{def.}}{=} G_2^{-1}(x,0)$ nært (x_0, y_0).

Videre: $F(x, \Phi(x)) = 0$

kj.r. $\Rightarrow [D_x F] + [D_y F][D\Phi] = 0$

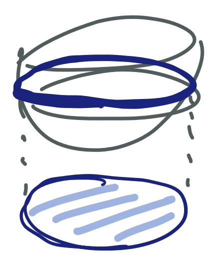
$\Rightarrow [D\Phi] = -[D_y F]^{-1}[D_x F]$ i $(x, \Phi(x))$.

13.1 – 13.2 Lokale og globale ekstrema

$F : U \subset \mathbb{R}^n \to \mathbb{R}$ kont. på kompakt $K \subset \bar{U}$

$\Rightarrow \exists\, x_0 \in K : F(x_0) = \max_{x \in K} F(x)$.

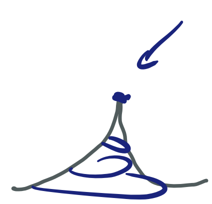
Hvordan finne x_0 / maks?



(i) x_0 kan tilhøre randen til K

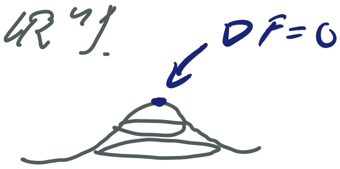
- Må sjekke $F(x)$; $x \in \partial K$.

(ii) x_0 kan være et punkt der F ikke er deriverbar.



- Må sjekke singulære punkter.

(iii) Ellers må $DF(x_0) = 0$ (i $\mathbb{R}^n$).



Bevis (for at (iii) er eneste andre mulighet)

x_0 indre punkt i $\bar{U}$, F deriverbar i x_0

$$\Longrightarrow \underbrace{F(x_0+h) - F(x_0)}_{(*)} = DF(x_0)\cdot h + |h|\underbrace{\varepsilon(h)}_{\substack{\to 0 \\ \text{da } h\to 0.}}$$

$F(x_0)$ lok. maks $\overset{\text{def.}}{\Longleftrightarrow} \exists\, \delta;\ (*) \le 0 \quad \forall\, |h| < \delta.$

Si $\boxed{\frac{\partial F}{\partial x_j}(x_0) \neq 0}$. Velg $h = (0,\dots,0,h_j,0,\dots,0)$;

$|\varepsilon(h)| \le \frac{1}{2}\left|\frac{\partial F}{\partial x_j}(x_0)\right|$ fikset fortegn, og $\neq 0$

$$\Longrightarrow \underbrace{\frac{F(x_0+h) - F(x_0)}{h_j}}_{\substack{\text{Veksler tegn med } h_j \\ \text{(eller helt lik 0)}}} = \overbrace{\frac{\partial F}{\partial x_j}(x_0) \pm \underbrace{\varepsilon(h)}_{\le \frac{1}{2}\left|\frac{\partial F}{\partial x_j}(x_0)\right|}}^{\text{fikset fortegn, og } \neq 0}$$

Motsigelse $\Longrightarrow \frac{\partial F}{\partial x_j}(x_0) = 0 \quad \forall\, j = 1, \dots, n$

$$\Longrightarrow DF(x_0) = 0$$

Obs! $\nabla F = 0$ nødvendig for lok. maks/min, men ikke tilstrekkelig.

kritisk punkt

stasjonært punkt

Si $F \in C^2(U, \mathbb{R})$, $\nabla F(x_0) = 0$.

$$\implies F(x_0 + h) - F(x_0) = \frac{1}{2} D^2 F(c)(h,h)$$

$$= \begin{bmatrix} h \end{bmatrix} \begin{bmatrix} D^2 F(c) \end{bmatrix} \begin{bmatrix} h \end{bmatrix}$$

kvadratisk form.

Hessematrise