- Generelt er curl(F) et lokalt mål på rotasjon, mens sirkulasjon er et globalt mål.

Egenskaper til curl og div ($\mathbb{R}^3$)

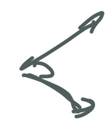
- $\overbrace{\operatorname{div}(\underbrace{\operatorname{curl}(\underbrace{F}_{\mathbb{R}^3})}_{\mathbb{R}^3})}^{\mathbb{R}} = 0$ i $\mathbb{R}$

- $\underbrace{\operatorname{curl}(\overbrace{\nabla \underbrace{\phi}_{\mathbb{R}}}^{\mathbb{R}^3})}_{\mathbb{R}^3} = 0$ i $\mathbb{R}^3$

Helmholtz:

$F: \mathbb{R}^3 \to \mathbb{R}^3$

$F = \underbrace{\nabla \phi}_{\text{rotasjonsfri}} + \underbrace{\nabla \times \psi}_{\text{divergens-fri}}$



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$$F(x,y,z) = (x\cos(y^2),\ z - x^2 y \sin(y^2),\ y)$$

(a) Finn curl(F).

(b) Beregn $\int_\gamma F \cdot T\, ds$ for

$$\gamma: t \mapsto (\sin(t),\ \sin(2t),\ t(\pi - 2t))$$

$t \in [0, \frac{\pi}{2}]$

Løsn. (a) $\operatorname{curl}(F) = \nabla \times F$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

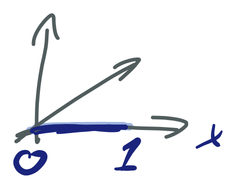
$$= \left(1 - 1, 0 - 0, -2xy\sin(y^2) + 2xy\sin(y^2)\right)$$

$$= (0, 0, 0)$$ F rotasjonsfri

(b) $\operatorname{curl}(F) = 0 \iff F$ konservativ

$\mathbb{R}^3$ enkeltsam.

$\iff \int_\gamma F \cdot T \, ds$ veiuavhengig



$\gamma(0) = (0,0,0)$, $\gamma(\frac{\pi}{2}) = (1, 0, 0)$

Velg $\tilde{\gamma}(x) = (x, 0, 0)$, $0 \leq x \leq 1$.

$$\boxed{F(x,y,z) = \left(x\cos(y^2), z - x^2 y \sin(y^2), y\right)}$$

$$\int_\gamma F \cdot T \, ds \underset{F \text{ kons}}{=} \int_{\tilde{\gamma}} F \cdot T \, ds = \int_0^1 F(\tilde{\gamma}(x)) \cdot \tilde{\gamma}'(x) \, dx$$

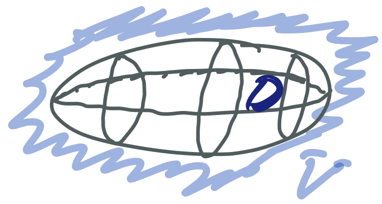
$$= \int_0^1 (x, 0, 0) \cdot (1, 0, 0) \, dx = \frac{1}{2}.$$

Alt. $\phi(x,y,z) = yz + \frac{x^2}{2}\cos(y^2)$ potensial,

dvs $\nabla\phi = F$.

$$\Rightarrow \int_r F\cdot T\,ds = \phi(1,0,0) - \phi(0,0,0) = \frac{1}{2}. \qquad \#$$

16.4 Divergensteoremet (Gauß' setning)



$D \subset \mathbb{R}^3$ kompakt, (lukket, begrenset)

med orienterbar rand ∂D

gitt ved en glatt flate $\left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right| \neq 0.$

$F \in C^1(V, \mathbb{R}^3)$, V åpen med $D \subset V$.

Da gjelder:

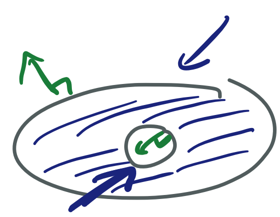
$$\boxed{\iint_{\partial D} F\cdot N\,d\sigma = \iiint_D \operatorname{div}(F)\,dV}$$

'Fluksen til F ut av ∂D'

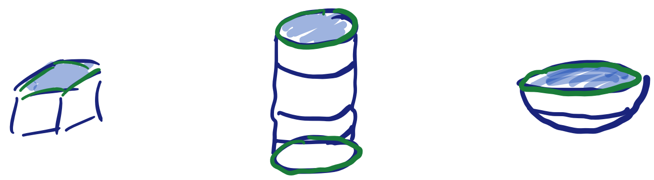
(N peker ut av D)

(i) $$N = \pm\frac{\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}}{\left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right|}, \quad d\sigma = \left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right| dt\,ds$$

(ii) ∂D kan bestå av flere deler



(iii) ∂D kan være stykkevis glatt:



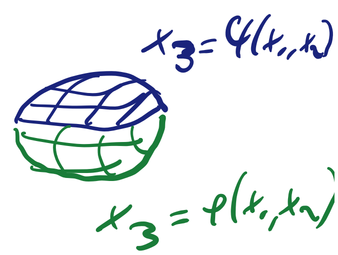
Bevis Likt Green's, for domener av typen $D = \{(x_1, x_2, x_3) : \varphi_k(x_i, x_j) \le x_k \le \psi_k(x_i, x_j)\}$

$(x_i, x_j) \in A$, $i, j, k \in \{1, 2, 3\}$, $i \neq j \neq k \neq i$.

Skriv $F = \underbrace{(F_1, 0, 0)}_{G_1} + \underbrace{(0, F_2, 0)}_{G_2} + \underbrace{(0, 0, F_3)}_{G_3}$

og vis at $$\boxed{\iint_{\partial D} G_j \cdot N \, d\sigma = \iiint_D \frac{\partial F_j}{\partial x_j} \, dV}$$

$$\overset{\sum_{j=1}^{3}}{\Longrightarrow} \quad \iint_{\partial D} F \cdot N \, d\sigma = \iiint_D \operatorname{div}(F) \, dV.$$



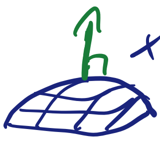
Viser for $j = 3$:

...

$$\iiint_D \frac{\partial F_3}{\partial x_3}\, dV = \iint_A \left(\int_{\varphi(x_1,x_2)}^{\psi(x_1,x_2)} \frac{\partial F_3}{\partial x_3}\, dx_3 \right) dx_1\, dx_2$$

$$= \iint_A \left(\underline{F_3(x_1,x_2,\psi(x_1,x_2))} - F_3(x_1,x_2,\varphi(x_1,x_2)) \right) dx_1\, dx_2$$

fund. setn.



$x_3 = \psi(x_1,x_2)$

'Toppen' $\quad \{ \underbrace{(x_1,x_2,\psi(x_1,x_2))}_{r(x_1,x_2) \text{ på grafform}} : (x_1,x_2) \in A \}$

med tangentene $\frac{\partial r}{\partial x_1} = \left(1, 0, \frac{\partial \psi}{\partial x_1}\right)$, $\frac{\partial r}{\partial x_2} = \left(0, 1, \frac{\partial \psi}{\partial x_2}\right)$

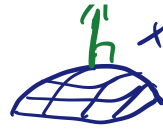
og $\frac{\partial r}{\partial x_1} \times \frac{\partial r}{\partial x_2} = \left(-\frac{\partial \psi}{\partial x_1}, -\frac{\partial \psi}{\partial x_2}, 1\right)$

$\vec n$ peker opp

$$\iint_A F_3(x_1,x_2,\psi(x_1,x_2))\, dx_1\, dx_2$$

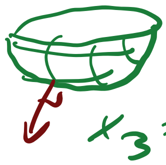
$$= \iint_A (0,0,F_3)\Big|_{x_3=\psi(x_1,x_2)} \cdot \left(-\frac{\partial \psi}{\partial x_1}, -\frac{\partial \psi}{\partial x_2}, 1\right) dx_1\, dx_2$$

$$= \iint G_3 \cdot N \, d\sigma \quad \text{på}$$



$x_3 = \varphi(x_1, x_2)$

på 'bunnen', likts



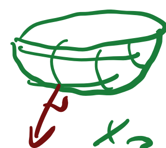
$x_3 = \varphi(x_1, x_2)$

$$-\iint_A F_3(x_1, x_2, \varphi(x_1, x_2)) \, dx_1 \, dx_2$$

$$= \iint_A (0, 0, F_3) \cdot \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, -1 \right)$$

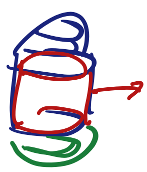
$F_3 \cdot (-1)$

$$= \iint G_3 \cdot N \, d\sigma \quad \text{på}$$



$x_3 = \varphi(x_1, x_2)$

• Hva hvis det finnes en vertikal del av ∂D?



$N = (N_1, N_2, 0)$

$\perp \; (0, 0, F_3)$.

$$\text{Så} \quad \iiint_D \frac{\partial F_3}{\partial x_3} \, dV = \iint_{\partial D} G_3 \cdot N \, d\sigma$$

$$\sum_{j=1}^{3} \Rightarrow \boxed{\iiint_D \operatorname{div}(F) \, dV = \iint_{\partial D} F \cdot N \, d\sigma}$$

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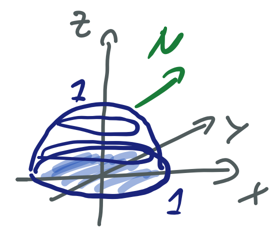
$F(x,y,z) = (x^3 z,\ y^3 z,\ x^2+y^2)$



$D = \{x^2+y^2+z^2 \le 1,\ z \ge 0\}$

Beregn fluksen ut av toppen til D:

$T = \{x^2+y^2+z^2 = 1,\ z \ge 0\}$.

Løsn. Bunn: $B = \{x^2+y^2 \le 1,\ z = 0\}$

$\partial D = T \cup B$ stykkevis glatt flate, $F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$

Kan bruke div. setn.: $\boxed{\iint_{\partial D} F \cdot N \, d\sigma = \iiint_D \operatorname{div}(F) \, dV}$

$\Rightarrow$ Fluks (ut av T): $\iint_T F \cdot N \, d\sigma = \underbrace{\iiint_D \operatorname{div}(F)\, dV}_{I} - \underbrace{\iint_B F \cdot N \, d\sigma}_{II}$

(I) $\operatorname{div}(F) = 3x^2 z + 3y^2 z + 0 = 3z(x^2+y^2)$.

Ii: $\iiint_D \operatorname{div}(F)\, dV = 3\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 \underbrace{\rho\cos\phi}_{z}\ \underbrace{\rho^2 \sin^2\phi}_{x^2+y^2}\ \underbrace{\rho^2 \sin\phi \ d\rho\, d\theta\, d\phi}_{dV}$

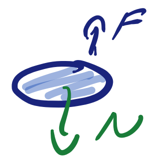
$x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\sin\phi$

$$= 6\pi \left(\underbrace{\int_0^{\pi/2} \cos\phi \sin^3\phi \, d\phi}_{\frac{\sin^4\phi}{4} \sim \frac{1}{4}} \right) \left(\underbrace{\int_0^1 \rho^5 \, d\rho}_{\frac{1}{6}} \right)$$

$z = \rho \cos\phi$

$$= 6\pi \cdot \frac{1}{4} \cdot \frac{1}{6} = \frac{\pi}{4}.$$

(2) $$\iint_B F \cdot N \, dS = \iint_B F \cdot (0,0,-1) \, dA$$



$$= \iint_{\substack{0 \le x^2+y^2 \le 1 \\ (z=0)}} F \cdot (0,0,-1) \, dA = -\int_0^{2\pi} \int_0^1 r^2 \underbrace{r \, dr \, d\theta}_{dA}$$

pol. koord.

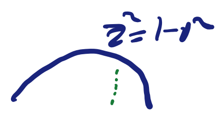
$$= -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}.$$

Svar: $$\iint_T F \cdot N \, dS = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}.$$

Alt. till (1) med cylinderkoord.

$x = r\cos\theta$	$0 \le \theta \le 2\pi$
$y = r\sin\theta$	$0 \le r \le 1$
$z = z$	$0 \le z \le \sqrt{1-r^2}$

$$dV = r \, dr \, d\theta \, dz$$

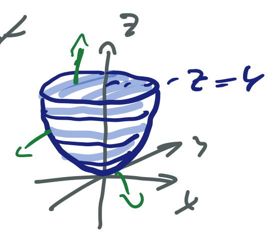


$$\iiint_D \underbrace{\operatorname{div}(F)}_{3z(x^2+y^2)}\, dV = 3\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} z\, r^2\, r\, dz\, dr\, d\theta$$

$$= 6\pi \int_0^1 r^3 \left[\frac{z^2}{2}\right]_0^{\sqrt{1-r^2}} dr = 3\pi \int_0^1 r^3(1-r^2)\, dr$$

$$= 3\pi\left[\frac{1}{4}-\frac{1}{6}\right] = \frac{\pi}{4}.$$

Eks Fluks ut av (over) flaten til legemet



$$D = \{(x,y,z) : \underbrace{x^2+y^2}_{r^2} \le z \le 4\}$$

for vektorfeltet $F:(x,y,z) \mapsto (x,y,3)$

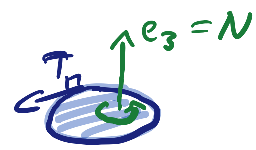
Divergensteoremet / Gauß' setning

$$\iint_{\partial D} F\cdot N\, dS = \iiint_D \underbrace{\operatorname{div}(F)}_{\sum_{i=1}^3 \frac{\partial F_i}{\partial x_i}}\, dV \qquad \sum_{i=1}^3 \frac{\partial F_i}{\partial x_i} = 1+1+0 = 2$$

$$\underset{\text{syl. koord.}}{=} 2\int_0^{2\pi}\int_0^2\left(\int_{r^2}^4 dz\right) r\, dr\, d\theta = 4\pi\int_0^2 (4-r^2)\, r\, dr$$

$$= 4\pi\left[2r^2 - \frac{r^4}{4}\right]_0^2 = 4\pi[8-4] = 16\pi.$$

16.5 Kelvin-Stokes' setning

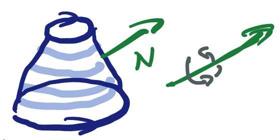


Husk Green i sirkulasjonsform:

$$\int_{\partial S} F\cdot T\,ds = \iint_S \underline{\text{curl}(F)}\,d\sigma$$

$$\text{curl}(F_1, F_2, 0)\cdot e_3 = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}$$

Nå, trekk flaten S opp til 3D



Stokes' er Green's for $F = (F_1, F_2, F_3)$, og S stykkevis glatt i 3D.

Teorem (Stokes')

(i) $r : A \subset \mathbb{R}^2 \longrightarrow S = r(A) \subset \mathbb{R}^3$

$(t,s) \longmapsto r(t,s)$ glatt flate

(kont. der. bar med $\left|\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s}\right| \neq 0$)

og injektiv.

orientert ved $N = \dfrac{\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s}}{|\text{—"—}|}$

med (orientert) rand $\partial S = r(\partial A)$.

(ii) $F \in C^2(\bar{V}, \mathbb{R}^3)$, $S \subset \bar{V}$ åpen.

Da er:

$$\boxed{\oint_{\partial S} F \cdot T \, ds = \iint_S \mathrm{curl}(F) \cdot N \, dS}$$

Bevis (for F og S C^2)

Skriv $S = \bigcup_{j=1}^{N} S_j$, der hver S_j er på grafformen:

$x = f(y,z)$, $y = f(x,z)$, eller $z = f(x,y)$.

Kan alltid gjøres da $\left|\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s}\right| \neq 0$,

ettersom

$$\frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s} = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{pmatrix}$$

$$= \left(\frac{\partial y}{\partial t}\frac{\partial z}{\partial s} - \frac{\partial z}{\partial t}\frac{\partial y}{\partial s},\ \frac{\partial z}{\partial t}\frac{\partial x}{\partial s} - \frac{\partial x}{\partial t}\frac{\partial z}{\partial s},\ \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial s} \right)$$

Minst én av disse skilt fra null i hvert punkt

Jakobideterminant til $\Phi : (t,s) \mapsto (x,y)$

$$D\phi = \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{pmatrix}.$$

Inverse f. teorem $\Longrightarrow$

$(t,s) \mapsto (x,y)$

C^2-bijeksjon

Nå kan skifte (t,s) mot (x,y):

$$r(t,s) = r(t(x,y), s(x,y)) = \tilde{r}(x,y) = (x, y, f(x,y))$$

(eller tilsv. for (x,z) alt. (y,z)).

Så anta $S_j = \{ z = f(x,y) : (x,y) \in A \}$

sånn at ∂S_j er gitt ved

$$r(\tau) = (x(\tau), y(\tau), f(x(\tau), y(\tau)))$$

Så:

$$\oint_\gamma F \cdot T\, ds = \oint_\gamma F_1 \dot{x} + F_2 \dot{y} + F_3 (f'_x \dot{x} + f'_y \dot{y})\, d\tau$$

$$= \oint_\gamma \underbrace{(F_1 + F_3 f'_x)}_{P} \underbrace{\dot{x}\, d\tau}_{dx} + \oint_\gamma \underbrace{(F_2 + F_3 f'_y)}_{Q} \underbrace{\dot{y}\, d\tau}_{dy}$$

Green

=

i 2D!

(x,y)

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = (*)$$

(x,y)

$F_j(x, y, f(x,y))$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(F_2 + F_3 f'_y \right) = \underline{\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z} f'_x}$$

$$+ \underline{\frac{\partial F_3}{\partial x} f'_y} + \cancel{\frac{\partial F_3}{\partial z} f'_x f'_y} + \cancel{F_3 f''_{yx}}$$

$$-\frac{\partial P}{\partial y} = -\frac{\partial}{\partial y}\left(F_1 + F_3 f'_x \right) = \underline{-\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z} f'_y}$$

$$\underline{-\frac{\partial F_3}{\partial y} f'_x} - \cancel{\frac{\partial F_3}{\partial z} f'_y f'_x} - \cancel{F_3 f''_{xy}}$$

Nå: normalen N til en graf $z = f(x,y)$

er $\dfrac{(-f'_x, -f'_y, 1)}{|-\,\cdot\,-|}$

$$\underline{N\,d\sigma = (-f'_x, -f'_y, 1)\,dx\,dy}$$

Grønne termer : $\left(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\right) f'_x + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) f'_y$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \cdot 1$$

$$= \operatorname{curl}(F) \cdot (-f'_x, -f'_y, 1) \quad !$$

Sei $\boxed{\oint_{\gamma} F \cdot T \, ds = \iint_{A} \operatorname{curl}(F) \cdot N \, d\sigma}$