Bevis (ii) ⇒ (i), se separat PDF på wikien.

---

Merk: Når $r(x) = (x, c_2, c_3, \dots, c_n)$ blir

$$F \cdot dr = F \cdot T\,ds = (F_1, F_2, \dots, F_n) \cdot \underbrace{(1, 0, \dots, 0)}_{r'(x)} dx$$

$$= F_1\,dx.$$

Dette motiverer notasjonen:

$$\boxed{\begin{aligned} dr = T\,ds &= F_1\,dx + F_2\,dy + F_3\,dz \quad &\text{i } \mathbb{R}^3 \\ &\phantom{=}\; F_1\,dx + F_2\,dy \quad &\text{i } \mathbb{R}^2 \end{aligned}}$$

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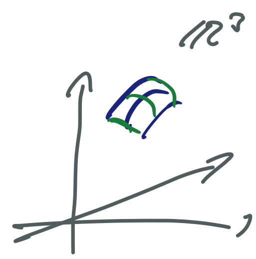
## 15.5 Flater og flateintegraler

Hva er en flate?




| kurver i $\mathbb{R}^2$ | flater i $\mathbb{R}^3$ | fremstillinger |
|---|---|---|
| $y = f(x)$ | $z = f(x,y)$ | graf |
| $f(x,y) = c$ | $f(x,y,z) = c$ | implisitt / nivåmengder (gitt $\nabla f \neq 0$) |
| $r(t) = (r_1(t), r_2(t))$ | ? | parametrisk |

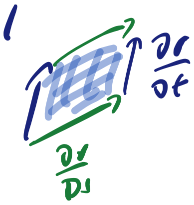
**Def** Glatte flater: "To familier av kurver som ikke er parallelle":

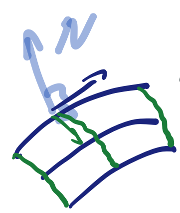
$$(s,t) \longmapsto \underline{\vec{r}(s,t) = (r_1(s,t), r_2(s,t), r_3(s,t))};$$

$s \mapsto r(s,t)$, $t \mapsto r(s,t)$ glatte kurver, og

$$\underbrace{\left| \frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \right|}_{\text{areal til}} \neq 0.$$

Merk: $\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}$ normal til flaten.

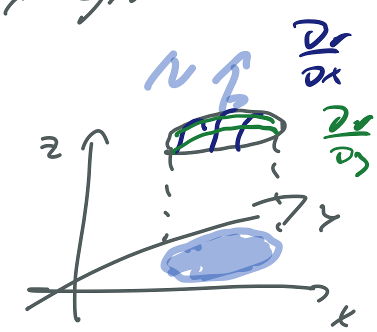



(N kont fordi $\partial_s r$ og $\partial_t r$ er kont.)

For en graf $z = f(x,y)$, $f \in C^1$, gir parametriseringen:

$$(x,y) \longmapsto \underbrace{(x, y, f(x,y))}_{r(x,y)}$$




$\frac{\partial r}{\partial x} = (1, 0, f_x(x,y))$: funksjon $\mathbb{R}^2 \to \mathbb{R}^3$

tangent i x-retning

$$\frac{\partial r}{\partial y} = (0, 1, f_y(x,y)) : \mathbb{R}^2 \to \mathbb{R}^3$$

tangent i $y$-retning

<u>normal</u>: $\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{pmatrix}$

$$= (-f_x, -f_y, 1) \neq (0,0,0)$$

---

<u>For en nivåflate</u>: $f(x,y,z) = c, \quad f \in C^2$

med $|\nabla f| \neq 0 \implies \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ eller $\frac{\partial f}{\partial z} \neq 0$

(i et gitt punkt)

Si: $\frac{\partial f}{\partial z} \neq 0 \overset{IFT}{\implies} \exists \, \phi \; C^1$:

$$f(x,y,z) = c \overset{\text{lokalt}}{\iff} z = \phi(x,y)$$

<u>lokalt</u> <u>en graf</u>!

<u>Parametrisering</u>: $(x,y) \mapsto (x, y, \phi(x,y))$

og likt som ovenfor: $\frac{\partial r}{\partial x} = (1, 0, \phi_x)$

Men: kan også

$$\frac{\partial r}{\partial y} = (0, 1, \phi_y)$$

løse ut $\phi_x, \phi_y$ i termer av $f$:

$$f(x,y,\phi(x,y)) = c \xrightarrow[\partial_y]{\partial_x} \begin{cases} \partial_x f + \partial_z f\, \partial_x \phi = 0 \\ \partial_y f + \partial_z f\, \partial_y \phi = 0 \end{cases}$$

Impl. der.

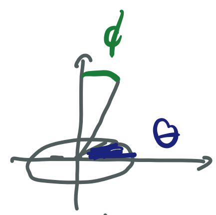
$$\boxed{\partial_x \phi = -\frac{\partial_x f}{\partial_z f}, \quad \partial_y \phi = -\frac{\partial_y f}{\partial_z f}}$$

$$\Rightarrow \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = (-\phi_x, -\phi_y, 1) = \left(\frac{f_x}{f_z}, \frac{f_y}{f_z}, 1\right)$$

$$= \frac{1}{f_z} \nabla f.$$


Eks. (i) Enhetssfæren (kuleskall): $x^2+y^2+z^2=1$

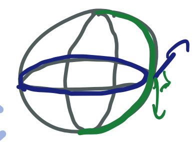
a) sfæriske koord. $(\phi, \theta)$




$$r(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$0 \le \phi \le \pi, \quad 0 \le \theta < 2\pi$$

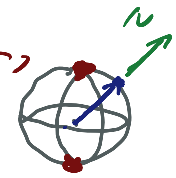
$$\begin{array}{cccc} & e_1 & e_2 & e_3 \\ \dfrac{\partial r}{\partial \phi} = & (\cos\phi\cos\theta, & \cos\phi\sin\theta, & -\sin\phi) \\ \dfrac{\partial r}{\partial \theta} = & (-\sin\phi\sin\theta, & \sin\phi\cos\theta, & 0) \end{array}$$



ortogonale!

$$\frac{\partial r}{\partial \phi} \times \frac{\partial r}{\partial \theta} = (\sin^2\phi\cos\theta,\ \sin^2\phi\sin\theta,\ \sin\phi\cos\phi)$$

$$= \underbrace{\sin\phi}_{0 \text{ for } \phi = 0 \text{ og } \phi = \pi}\ r(\phi,\theta) \quad // \ r \text{ selv!}$$





b) kartesiske koord. $\underbrace{x^2+y^2+z^2}_{f(x,y,z)} = 1$

Finner en lokal, f.eks.

$$\boxed{\left\{ z = \underbrace{\sqrt{1-x^2-y^2}}_{\phi(x,y)} : 0 \leq x^2+y^2 \leq 1 \right\}} \cup \left\{ z = -\sqrt{\cdots} \right\}$$

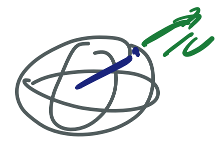
$$(x,y) \overset{r}{\longmapsto} (x,y,\underbrace{\sqrt{1-x^2-y^2}}_{z})$$

$$\frac{\partial r}{\partial x} = \Big(1,\ 0,\ \underbrace{\frac{-x}{\sqrt{\phantom{1}}}}_{\phi_x}\Big),\quad \frac{\partial r}{\partial y} = \Big(0,\ 1,\ \underbrace{\frac{-y}{\sqrt{\phantom{1}}}}_{\phi_y}\Big)$$

$$\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \frac{\nabla f}{f_z} = \frac{(2x, 2y, 2z)}{2z}$$

$$= \left(\frac{x}{\sqrt{1-x^2-y^2}},\ \frac{y}{\sqrt{1-x^2-y^2}},\ 1\right)$$

Igjen: normalen parallell til $r$ selv.




**Mange parameteriseringer mulige!**

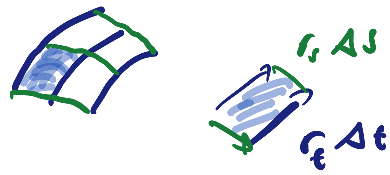
---

Øvelse: Plan $\underbrace{Ax + By + Cz}_{f(x,y,z)} = D$, $\|(A, B, C)\| \neq 0$. Parametriser.

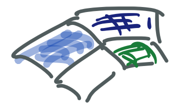
Finn tangentvektorer og normal.

Når kan planet parametriseres i $(x,y)$; i $(x,z)$; i $(y,z)$?

Nå: ønsker å integrere over en flate $S$.

Idé: 


$$\lim \sum |r_t \times r_s| \Delta s \Delta t$$



**Def.** (i) Arealet til en flate $S$ gitt ved

$r \in C^1(\bar{U}, \mathbb{R}^3)$, $\bar{U} \subset \mathbb{R}^2$, med $\left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| \neq 0$,

$s, t \in \bar{U}$, er

$$A(S) \overset{\text{def.}}{=} \iint_{\bar{U}} \underbrace{\left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| ds\, dt}_{d\sigma} = \iint_S d\sigma$$

(ii) <u>Flateintegralet av et skalarfelt</u> $f \in C(\bar{V}, \mathbb{R})$

over $S$ som ovenfor, $S \subset \bar{V}$, $\bar{V} \subset \mathbb{R}^3$, er
åpen

$$\iint_S f\, d\sigma = \iint_{\bar{U}} f(r(s,t)) \left|\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}\right| ds\, dt.$$

# Eks Oppg 4 15/8 2009

Beregn $A(S)$, der $S$ er delen av flaten $\{z = \ln(x^2+y^2)\}$ i første oktant, under planet $\{z=2\}$.

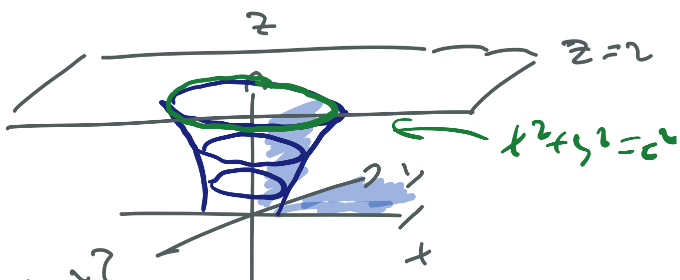
---

**Løsn.** Første oktant: $x, y, z \geq 0$.

$$\Rightarrow \quad 0 \leq z \leq 2. \quad \Rightarrow \quad \boxed{\begin{matrix} 0 \leq \ln(x^2+y^2) \leq 2 \\ x, y \geq 0 \end{matrix}}$$

$$\Leftrightarrow \quad \boxed{\begin{matrix} 1 \leq x^2+y^2 \leq e^2 \\ x, y \geq 0 \end{matrix}}$$

**Kartesiske koord.**




$$S = \{(x, y, \ln(x^2+y^2)) : x \geq 0,\ y \geq 0,\ 1 \leq x^2+y^2 \leq e^2\}$$

$$\left|\frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y}\right| = \left| \det \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & \frac{2x}{x^2+y^2} \\ 0 & 1 & \frac{2y}{x^2+y^2} \end{pmatrix} \end{array} \right|$$

$$= \Big\| \underbrace{\left(-\frac{2x}{x^2+y^2}, -\frac{2y}{x^2+y^2}, 1\right)}_{\text{kont på } S!} \Big\| = \sqrt{\frac{4(x^2+y^2)}{(x^2+y^2)^2}+1}.$$

pol. koord. $(a,\theta)$

$r$

$$d\sigma = \sqrt{\frac{4}{x^2+y^2}+1}\;dx\,dy = \sqrt{1+\frac{4}{a^2}}\;a\,da\,d\theta$$

$$\boxed{\begin{array}{l}(x,y)\leftrightarrow(r,\theta)\\ dx\,dy\\ = r\,dr\,d\theta\end{array}}$$

Så: $A(S) = \iint_S d\sigma = \int_0^{\frac{\pi}{2}}\int_1^e \sqrt{1+\frac{4}{a^2}}\;a\,da\,d\theta$

$1 \le a \le e$

$0 \le \theta \le \frac{\pi}{2}$

$$= \frac{\pi}{2}\int_1^e \sqrt{4+a^2}\,da = \dots \text{ (hyperbolsk var.bytte)}$$

---

Merk: Hvis vi bruker sylinderkoord direkte: $r(a,\theta) = (a\cos(\theta), a\sin(\theta), \ln(a^2))$

$$\Rightarrow \left|\frac{\partial r}{\partial a} \times \frac{\partial r}{\partial \theta}\right| = \sqrt{4+a^2}$$

$$d\sigma = \sqrt{4+a^2}\;da\,d\theta$$

---

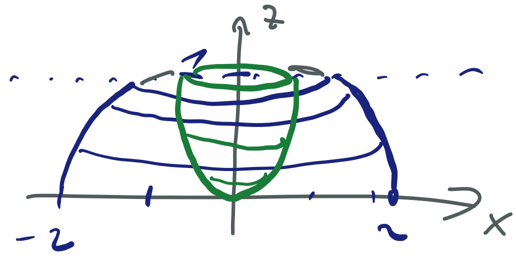
Eks opps 3 19.12 2006

$R$ avgrenset av $x^2+y^2+z^2=4$, $z=x^2+y^2$,

paraboloide

$z=0$ og $z=1$.

plan




c) Stjernestøv på

$\{x^2+y^2+z^2=4\}$-delen med tetthet $f(x,y,z)=z$.

Beregn mengden stjernestøv.

---

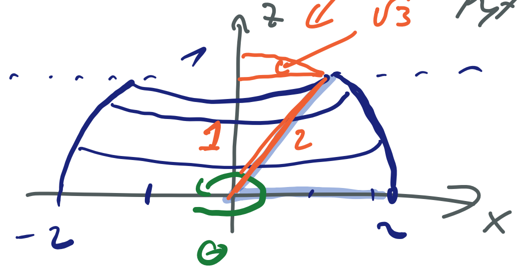
Løsn.

$$\rho=2$$

$$r(\phi,\theta)=(2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$$

sf. koord.

$$\boxed{2\cos(\phi)=1.}$$




$$\left|\frac{\partial r}{\partial\phi}\times\frac{\partial r}{\partial\theta}\right|$$

$$0\le\theta\le 2\pi$$

$$\frac{\pi}{3}\le\phi\le\frac{\pi}{2}$$

$$=\begin{vmatrix} e_1 & e_2 & e_3 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$=|(4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\cos\phi\sin\phi)|$$

$= 4\sin\phi.$

$$\iint_S f\, d\sigma = \int_0^{2\pi}\int_{\pi/3}^{\pi/2} \underbrace{2\cos\phi}_{z}\; 4\sin\phi\, d\phi\, d\theta$$

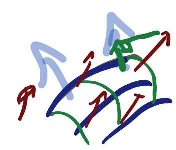
$$= 4\cdot 2\pi \sin^2\phi \Big|_{\pi/3}^{\pi/2} = 8\pi\left(1-\left(\frac{\sqrt{3}}{2}\right)^2\right) = 2\pi.$$

$$\frac{d}{d\phi}\sin^2\phi = 2\cos\phi\sin\phi$$

## 15.6 Fluks (flyt)

Fluksintegraler er et mål på flyten av et kont vektorfelt $F$ gjennom en flate $S$:

$$\boxed{\iint_S F\cdot \underbrace{N\, d\sigma}_{d\vec{S}} = \iint_S F\cdot d\vec{S}}$$



der $N$ er en enhetsnormal til $S$.

For en parametrisering $(s,t)\mapsto r(s,t)$:

$$\iint_D \vec{F}(r(s,t)) \cdot \pm\frac{\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}}{\cancel{\left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right|}}\;\underbrace{\cancel{\left|\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right|}\, ds\, dt}_{d\sigma}$$

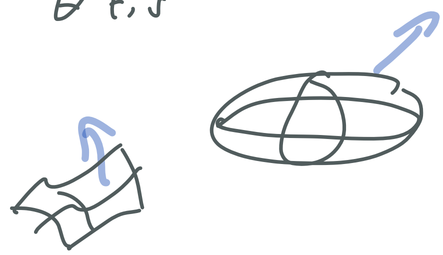
kontekstuelt!

**Def** En flate er orienterbar dersom det finnes en kont (rettet) normal i hvert punkt på $S$.

I vårt tilfelle tilsvarer dette:

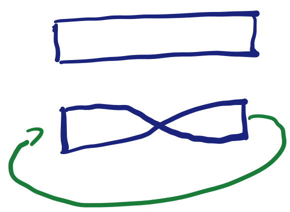
$\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t}$ er kont og $\neq 0$ $\forall t, s$

$+$ $(s,t) \mapsto r(s,t)$ injektiv.



Men: finnes ikke-orienterbare flater

Møbius bånd



Hvordan vil retningen til $N$ bli bevart?