$$= \cancel{2}\pi \int_0^1 \left[\frac{r^2}{\cancel{2}}\right]_{\sqrt{z}}^{\sqrt{4-z^2}} dz = \pi \int_0^1 \left((4-z^2)-z\right) dz = \frac{19\pi}{6}.$$

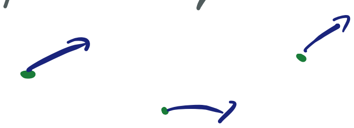
$4-z^2 \geq 0$

$0 \leq z \leq 1$

## 15.1 - 15.2 Vektor- og skalarfelt

**Def.** • Et vektorfelt er en funksjon/avbildning $U \subset \mathbb{R}^n \to \mathbb{R}^n$.

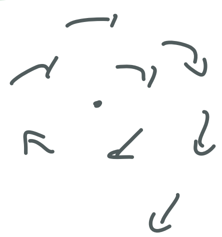
Til hvert punkt i $\mathbb{R}^n$



tildeler vi en vektor av samme dimensjon.

Typisk eksempel:

Værkartet



• Et skalarfelt er en avbildning $U \subset \mathbb{R}^n \to \mathbb{R}$.

Det vanligste eksemplet på vektorfelt er et gradientfelt.

**Def.** $F : U \overset{\text{åpen}}{\subset} \mathbb{R}^n \to \mathbb{R}^n$ gradientfelt $\overset{\text{def.}}{\iff}$

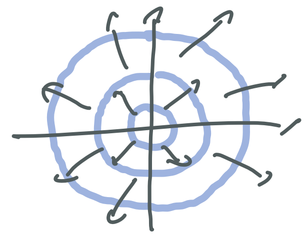
$F = \nabla f$, $f : U \subset \mathbb{R}^n \to \mathbb{R}$

Skalarfelt

**Eks.** $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = x^2 + y^2$

$F = \nabla f : (x,y) \mapsto (2x, 2y)$ er et gradientfelt.

**Def** Et vektorfelt $F$ er <u>konservativt</u> dersom det finnes en $\phi : \mathbb{R}^n \to \mathbb{R}$; $F = \nabla \phi$.



Funksjonen $\phi$ kalles <u>potential til $F$</u> (unik opp til en konstant).

<u>Når finnes en potential?</u>

- <u>Dersom</u> $\phi$ finnes, må $F = \left( \frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n} \right)$.

  Så, gitt $F$ er $C^1$, blir $\phi$ $C^2$, og

  $\partial x_i \underbrace{\partial x_j}\, \phi = \partial x_j \underbrace{\partial x_i}\, \phi$, $1 \leq i,j \leq n$, dvs

  $$\boxed{\frac{\partial F_j}{\partial x_i} = \frac{\partial F_i}{\partial x_j}}$$

  er et <u>nødvendig</u> vilkår for eksistens av en potential $\phi \in C^2$.

Hva er bra med konservative felt?

Som å finne anti-deriverte i integraler.

Først: kurvintegraler

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## 15.3 - 15.4 Kurvintegraler av felt

Husk at $r \in C^1(I, \mathbb{R}^n)$ med $|\dot{r}| \neq 0$ på $I$, $|v|$

har lengde
$$\boxed{\int_{r} ds = \int_{I} |\dot{r}(t)|\, dt}$$
, der

$$s(t) = \int_{t_0}^{t} |\dot{r}(\tau)|\, d\tau, \quad \text{så} \quad \frac{ds}{dt} = |\dot{r}(t)| \neq 0.$$

---

Def. $I \subset \mathbb{R}$, $U \subset \mathbb{R}^n$, $r \in C^1(I, \mathbb{R}^n)$:

$r(I) \subset \bar{U}$ og $|\dot{r}| \neq 0 \;\; \forall t \in I$.

Kurvintegralet av et skalarfelt $f \in C(U, \mathbb{R})$

langs kurven $r$ er:

$$\boxed{\int_{r} f\, ds = \int_{I} f(r(t))\, |\dot{r}(t)|\, dt}$$

**Eks. oppg. 4(a) 8/8 2005**

Beregn buelengde til $C = \left\{\left(\frac{1}{t}, \sqrt{2}\,t, \frac{t^3}{3}\right) : \frac{1}{2} \le t \le 2\right\}$

**Løsn** $\gamma : t \longmapsto \left(\frac{1}{t}, \sqrt{2}\,t, \frac{t^3}{3}\right)$ er $C^1$ på $(0,\infty)$ ↑!

og på $\frac{1}{2} \le t \le 2$ med

$$|\dot\gamma(t)| = \left|\left(-\frac{1}{t^2}, \sqrt{2}, t^2\right)\right| = \underbrace{\sqrt{\frac{1}{t^4} + 2 + t^4}}_{\left(\frac{1}{t^2}+t^2\right)^2}$$

$$= \frac{1}{t^2} + t^2 > 0 \text{ på } (0,\infty).$$

Buelengde: $\displaystyle\int_\gamma ds = \int_{1/2}^{2} |\dot\gamma(t)|\,dt = \int_{1/2}^{2} \left(\frac{1}{t^2}+t^2\right)dt$

$$= \left[-\frac{1}{t} + \frac{t^3}{3}\right]_{\frac{1}{2}}^{2} = -\frac{1}{2} + \frac{8}{3} + 2 - \frac{1}{3\cdot 8} = \frac{33}{8}.$$

**Eks. oppg. 1 4/8 2008** (modifisert)

$$\gamma : t \longmapsto \underbrace{(x,y,z)}_{x(t)} = \left(t, \frac{t^2}{\sqrt{2}}, \frac{t^3}{3}\right), \quad 0 \le t \le 2.$$

Bestem $\int_\gamma x \, ds$.

---

**Løsn.** $\gamma$ glatt med $|\dot\gamma(t)| = |(1, \sqrt{2}t, t^2)|$

$$= \sqrt{\underbrace{1 + 2t^2 + t^4}_{(1+t^2)^2}} = 1 + t^2 > 0 \quad \forall t \in \mathbb{R}.$$

$\Rightarrow \; ds = (1+t^2)\,dt$ buelengdeelementet

$$\int_\gamma x \, ds = \int_0^2 t(1+t^2)\,dt = \left[\frac{t^2}{2} + \frac{t^4}{4}\right]_0^2 = 6.$$

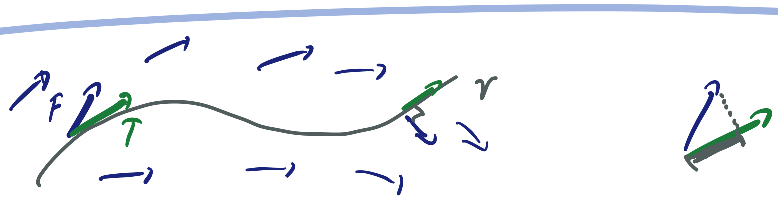
---

**Def.** Kurveintegralet av et kontinuerlig vektorfelt

$F : \overset{\text{åpen}}{\bar{U}} \subset \mathbb{R}^n \to \mathbb{R}^n$ langs en glatt kurve

$\gamma : I \to \bar{U}$, er:
$\quad t \mapsto \gamma(t)$

$$\boxed{\int_\gamma \vec{F} \cdot d\vec{r} \overset{\text{def.}}{=} \int_\gamma \vec{F} \cdot \underbrace{\vec{T}\,ds}_{d\vec{r}} = \int_I F(\gamma(t)) \cdot \dot\gamma(t)\,dt}$$



$F\cdot T$ projeksjonen av $F$ i retning $T$, der $|T|=1$.

Merk: $\vec{T}\,ds = \frac{\dot{\vec{\gamma}}(t)}{\cancel{|\dot\gamma(t)|}}\,|\dot\gamma(t)|\,dt = \dot{\vec{\gamma}}(t)\,dt$.

Prop $\int_\gamma F\cdot dr$ uavhengig av parameterisering.

Bevis $\int_\gamma F\cdot dr = \int_a^b F(\gamma(t))\cdot\dot\gamma(t)\,dt = (*)$

$\gamma(a)=\vec{x}_0$, $\gamma(b)=\vec{y}_0$

La $s(t) = \int_a^t |\dot\gamma(\tau)|\,d\tau$, $s(a)=0$, $s(b)=L(\gamma)$.

$\frac{ds}{dt} = |\dot\gamma(t)|$, $\tilde\gamma(s) = \gamma(t(s))$

$\Rightarrow \boxed{\frac{d\gamma}{dt} = \frac{d\tilde\gamma}{ds}\frac{ds}{dt}}$ Variabelsubst. $\Rightarrow \boxed{\dot\gamma(t)\,dt = \tilde\gamma'(s)\,ds}$

$(*) = \int_0^L F(\underbrace{\tilde\gamma(s)}_{\gamma(t(s))})\cdot\tilde\gamma'(s)\,ds$, uavhengig av $t$.

Buelengdeparameteren er unik. $\Rightarrow$ entydig def. (naturlig param.)

**Teorem** $F \in C^1(U, \mathbb{R}^n)$ konservativt,

$\gamma : [a,b] \to U$ glatt,

$$\Rightarrow \boxed{\int_\gamma F \cdot dr = \phi(\gamma(b)) - \phi(\gamma(a))}$$

avhengig kun av endpunktet

---

**Bevis** $$\int_\gamma F \cdot dr = \int_\gamma F \cdot T \, ds = \int_a^b F(\gamma(t)) \cdot \dot\gamma(t) \, dt$$

$\nabla\phi = F$ konservativt

$$= \int_a^b \nabla\phi(\gamma(t)) \cdot \dot\gamma(t) \, dt \overset{\text{kj.reg.}}{=} \int_a^b \frac{d}{dt} \phi(\gamma(t)) \, dt$$

$$\overset{\text{fund.teor.}}{=} \phi(\gamma(b)) - \phi(\gamma(a)).$$

$\square$

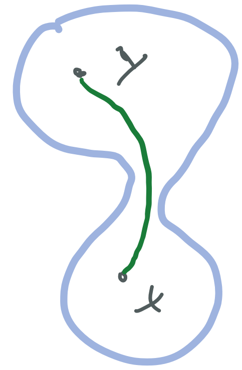
---

Hvordan vite om $\phi$ eksisterer? ($F$ konservativt)

**Def.** (i) $U \subset \mathbb{R}^n$ sammenhengende dersom

$\forall x, y \in \bar{U} \quad \exists \gamma \in C([0,1], \bar{U});$

$$\boxed{\gamma(0) = x, \quad \gamma(1) = y.}$$

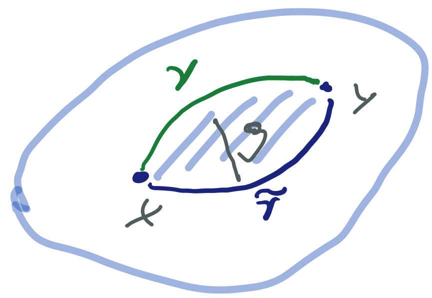



(ii) $\bar{U}$ er enkelt sammenhengende dersom

$\forall \gamma, \tilde{\gamma} \in C([0,1], \bar{U}), \ \forall x, y \in \bar{U},$

$\gamma(0) = \tilde{\gamma}(0) = x,$

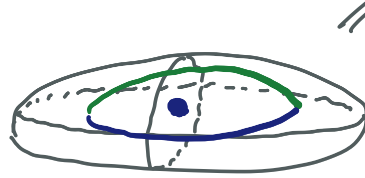
$\gamma(1) = \tilde{\gamma}(1) = y,$



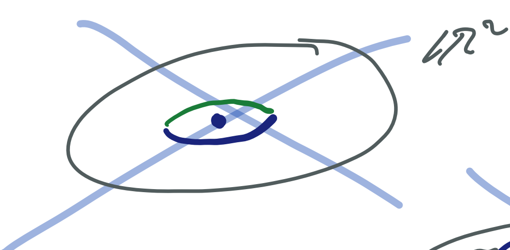

$\exists g : C([0,1] \times [0,1], U), \ (t, \lambda) \to g(t, \lambda);$

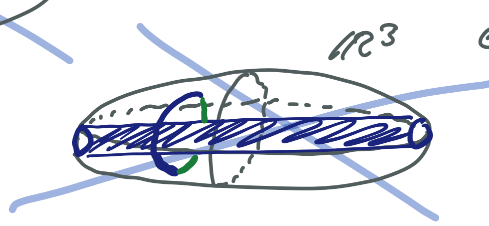
$$\boxed{g(t,0) = \gamma(t), \quad g(t,1) = \tilde{\gamma}(t).}$$

Finnes ingen 'gjennomgående hull'




enkelt sammenhengende



**Teorem** $U \subset \mathbb{R}^n$ åpen, enkelt sammenhengende, $F \in C^1(U, \mathbb{R}^n)$. Da er følgende vilkår ekvivalente:

(i) $F$ konservativt

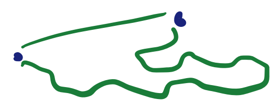
(ii) $\partial_{x_j} F_i = \partial_{x_i} F_j$, $1 \le i, j \le n$

$\gamma(a) = \gamma(b)$.

(iii) $\oint_\gamma F \cdot dr = 0$ $\forall$ lukket kurve $\gamma$ i $U$.

(iv) $\int_a^b F \cdot dr = \phi(\gamma(b)) - \phi(\gamma(a))$ (Det finnes en $\phi$)

'veiuavhengig'



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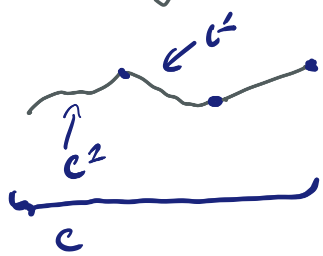
**Merk:** Tilstrekkelig at $\gamma$ er stykkevis glatt, dvs




Gjelder alle kurveintegraler!

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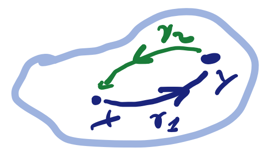
**Bevis** (i) $\Rightarrow$ (ii) allerede bevist

(i) $\Rightarrow$ (iv) — " —

(iv) $\Rightarrow$ (iii) Trivielt, fordi $\gamma(a) = \gamma(b)$ for en lukket kurve $\underset{(iv)}{\Rightarrow}$ $\phi(\gamma(a)) - \phi(\gamma(b)) = 0$.

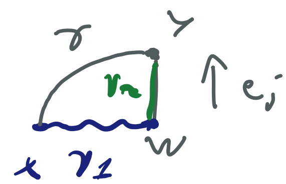
(iii) ⇒ (iv)



$$\int_{\gamma_1} F\cdot dr = \underbrace{\oint_{\gamma} F\cdot dr}_{0} - \int_{\gamma_2} F\cdot dr = -\int_{\gamma_2} F\cdot dr$$

Så: veiuavhengig

(iv) ⇒ (i) Fikser et punkt $x \in \overline{\Omega}$, la $\gamma$ være en kurve fra $x$ til $y \in \overline{\Omega}$.



$$\text{La } \phi(y) \overset{def.}{=} \underbrace{\int_{\gamma} F\cdot dr}_{\text{veiuavhengig}}$$

$w_j = x_j$

$$\frac{\partial}{\partial y_j}$$

$$\overset{(iv)}{=} \underbrace{\cancel{\int_{\gamma_1} F\cdot dr}}_{\text{konst. i } e_j} + \underbrace{\int_{\gamma_2} F\cdot dr.}_{\text{varierer i } e_j\text{-retn.}}$$

$x_j < y_j$

$$\gamma_2 : (w_1, w_2, \ldots, \underset{\substack{\uparrow \\ j}}{t}, \ldots, w_n), \quad t \in [x_j, y_j]$$

$$\Rightarrow \frac{\partial \phi}{\partial y_j} = \frac{\partial}{\partial y_j} \int_{x_j}^{y_j} F(\gamma(t)) \cdot \underbrace{\gamma'(t)}_{(0,0,\ldots,1,0,\ldots,0)}\, dt$$

$$= \frac{\partial}{\partial y_j} \int_{x_j}^{y_j} F_j(\gamma(t))\, dt \underset{\text{Fund.teo.}}{=} F_j(\gamma(y_j)) = \underline{\underline{F_j(y)}}$$

Sant for $j = 1, 2, \dots, 4 \implies \boxed{\nabla \phi = F.}$

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Eks. oppg. 4 1/8 2002

$F(x, y, z) = (-y, x, e^z)$

$\gamma : t \longmapsto (\underset{x}{\cos(t)}, \underset{y}{\sin(t)}, \underset{z}{\tfrac{t}{2}}), \quad t \in [0, 4\pi]$

Beregn $\int_\gamma F \cdot T \, ds = \int_\gamma F \cdot dr$

---

Løsn. $\int_\gamma F \cdot T \, ds$

$$= \int_0^{4\pi} \underbrace{(-\sin(t), \cos(t), e^{t/2})}_{F(\gamma(t))} \cdot \underbrace{(-\sin(t), \cos(t), \tfrac{1}{2})}_{\dot{\gamma}(t)} \, dt$$

$T = \frac{\dot{r}}{|\dot{r}|}, \quad ds = |\dot{r}| \, dt$

$$= \int_0^{4\pi} (\underbrace{\sin^2(t) + \cos^2(t)}_{1} + \frac{e^{t/2}}{2}) \, dt = 4\pi + e^{2\pi} - 1.$$

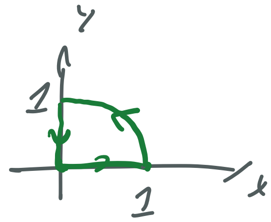
---

Eks. $F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \to (x^2, xy)$

$\gamma_1 : x \mapsto (x,0),\ x \in [0,1]$

$\gamma_2 : \theta \mapsto (\cos(\theta), \sin(\theta)),\ \theta \in [0, \frac{\pi}{2}]$

$\gamma_3 : y \mapsto (0,y),\ y$ fra 1 til 0!




$$\oint_\gamma F\cdot T\,ds = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3}\right) F\cdot T\,ds.$$

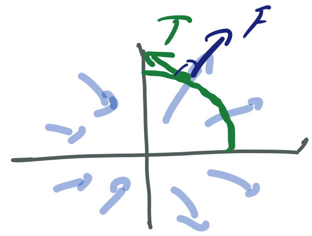
$$\int_{\gamma_1} F\cdot T\,ds = \int_0^1 \underbrace{(x^2,0)}_{F(\gamma_1(x))}\cdot\underbrace{(1,0)}_{\dot\gamma_1(x)}\,dx = \int_0^1 x^2\,dx = \frac{1}{3}.$$

$$\int_{\gamma_2} F\cdot T\,ds = \int_0^{\pi/2} \underbrace{(\cos^2(\theta), \cos(\theta)\sin(\theta))}_{F(\gamma_2(\theta))}\cdot\underbrace{(-\sin(\theta), \cos(\theta))}_{\dot\gamma_2(\theta)}\,d\theta$$

$$= 0.$$

Hvorfor?

$F(x,y) = (x^2, xy) = x\,(x,y)$




$$\int_{\gamma_3} F\cdot T\,ds = \int_1^0 (0,0)\cdot(0,1)\,dy = 0$$

! !

Så $\oint_\gamma F\cdot T\,ds = \frac{1}{3}.$ (F ikke konservativt!)

Sjekk at $\oint_\gamma F\cdot dr = 0$