téorem

12.3 Partielt derivate

For funksjoner $f: \mathbb{R} \to \mathbb{R}$ er

(i) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, og

(ii) $f(x+h) = f(x) + f'(x)h + h\underbrace{\varepsilon(h)}_{\to 0 \text{ da } h \to 0.}$

ekvivalente def. av den derivate.

I høyere dim. gir de to forskjellige derivate.

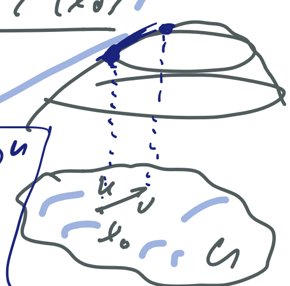
I høyere dim. gir disse to forskjellige begreper av den deriverte.

Def. La $F : U \subset \mathbb{R}^n \to \mathbb{R}$, $x_0 \in U$,
$v \in \mathbb{R}^n$, $|v| = 1$.

(i) Den retningsderiverte av F i punktet x_0 i retning v er:

$$\boxed{\partial_v F(x_0) \overset{\text{def.}}{=} \lim_{h \to 0} \frac{F(x_0 + hv) - F(x_0)}{h}}$$

(dersom grensen eksisterer) $\boxed{h \in \mathbb{R},\ v \in \mathbb{R}^n \quad \partial_v F(x_0) \in \mathbb{R}}$



'hvor raskt vokser F i retning v fra x_0?'

(Kan også skrive $\frac{\partial F}{\partial v}$, $D_v F$, F'_v, ...)

- De enkleste retningsderiverte er når $v = e_i$, (basisvektor i $\mathbb{R}^n$). Disse kalles **partielt deriverte**.

Eks. $F(x_1, x_2) = x_1^2 \sin(x_2)$

$$v = (1,0) \Longrightarrow \partial_v F(x_1, x_2) = \partial_{x_1} F(x_1, x_2)$$
$$= 2x_1 \sin(x_2).$$
$$v = (0,1) \Longrightarrow \partial_v F(x_1, x_2) = \partial_{x_2} F(x_1, x_2)$$
$$= x_1^2 \cos(x_2).$$

'vanlig' derivert

$\partial_{x_j} F$ skrives også

$\frac{\partial F}{\partial x_j}$, F'_{x_j}, $D_{x_j} F$, ...

$v = (1,0) \Rightarrow \partial_v F(x_1, x_2) = \partial_{x_1} F(x_1, x_2)$

$= 2x_1 \sin(x_2).$

$v = (0,1) \Rightarrow \partial_v F(x_1, x_2) = \partial_{x_2} F(x_1, x_2)$

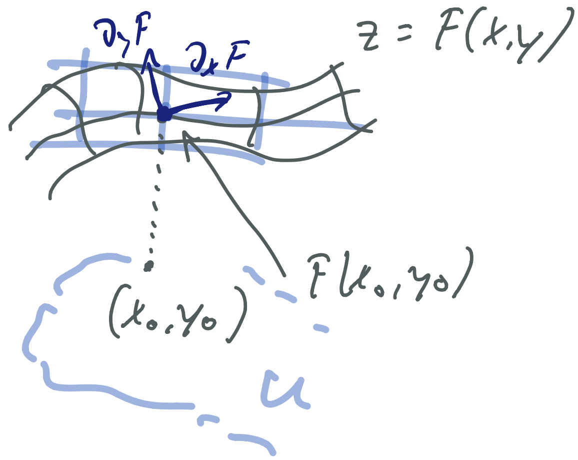
$= x_1^2 \cos(x_2).$

'vanlig' derivert

$\partial_{x_j} F$ skrives også

$\frac{\partial F}{\partial x_j}, \; F'_{x_j}, \; D_{x_j} F, \ldots$

$\partial_{x_j} F$ er forandringen i retning x_j



Tangentplanet

$$\boxed{z = \underbrace{F(x_0,y_0)}_{z_0 \text{ (fiksert)}} + \overbrace{(x-x_0)F'_x(x_0,y_0)}^{\text{forandr. i } x} + \underbrace{(y-y_0)F'_y(x_0,y_0)}_{\text{forandr. i } y}}$$

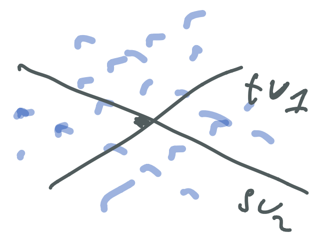
La $t = x - x_0 \iff x = x_0 + t$

$s = y - y_0 \iff y = y_0 + s$

$\implies z = z_0 + t\,F'_x(x_0,y_0) + s\,F'_y(x_0,y_0)$

Altså: $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ F'_x(x_0,y_0) \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ F'_y(x_0,y_0) \end{bmatrix}$$

Plan gjennom (x_0, y_0, z_0) i retning $\underbrace{(1, 0, F'_x)}_{v_1}$ og $\underbrace{(0, 1, F'_y)}_{v_2}$



Kan også skrive $\boxed{\quad\quad}$ som:

$$\underbrace{(x - x_0, y - y_0, z - z_0)}_{\text{span}(v_1, v_2)} \cdot \underbrace{(F'_x, F'_y, -1)} = 0$$

$\implies$ normal til TP!

Sjekk: $(1, 0, F'_x) \cdot (F'_x, F'_y, -1) = 0$

$(0, 1, F'_y) \cdot (F'_x, F'_y, -1) = 0$

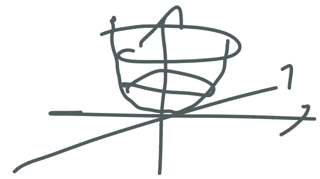
Kortversjon, gradient og eksempel

Def. For $F : U \subset \mathbb{R}^n \to \mathbb{R}$ part. der.bar er

$$\nabla F \overset{\text{def.}}{=} \left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}\right) \quad \text{gradienten til } F.$$

$\boxed{\text{Obs! } \nabla F : U \subset \mathbb{R}^n \to \mathbb{R}^n \text{ !}}$

Eks. $F(x,y) = x^2 + y^2$



Gradient: $\nabla F(x,y) = (2x, 2y)$

$\mathbb{R}^2 \to \mathbb{R}^2$

skål: $z = x^2 + y^2$

TP gjennom $(x_0, y_0, z_0) = (x_0, y_0, \underbrace{x_0^2 + y_0^2}_{F(x_0,y_0)})$:

$$\boxed{z = z_0 + (x - x_0, y - y_0) \cdot \nabla F(x_0, y_0)}$$

$v_1 = (1, 0, 2x_0)$, $v_2 = (0, 1, 2y_0)$, $N = (2x_0, 2y_0, -1)$

I punktet $(x_0, y_0) = (0, 0)$:

$z_0 = x_0^2 + y_0^2 = 0.$

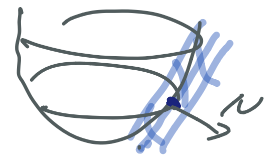
$$z = 0 + (x - 0, y - 0) \cdot \overbrace{(2\cdot 0, 2\cdot 0)}^{\nabla F(0,0)} = 0$$

$v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $N = (0, 0, -1)$

I punktet $(x_0, y_0) = (1, 1)$

$z_0 = 1^2 + 1^2 = 2$

$$z = 2 + (x-1, y-1)\cdot\underbrace{(2\cdot 1, 2\cdot 1)}_{\nabla F(1,1)}$$



$$= 2 + 2(x-1) + 2(y-1)$$

$v_1 = (1,0,2)$, $v_2 = (0,1,2)$, $N = (2,2,-1)$

Deriverbarhet (forts. def.)

Def. $F : \underset{\text{åpen}}{U} \subset \mathbb{R}^n \to \mathbb{R}$ er

(ii) deriverbar i $x_0 \in U$ dersom

$$F(x_0 + \underset{\text{vektor}}{h}) = \underbrace{F(x_0)}_{\text{fiksert}} + \underbrace{h\cdot\nabla F(x_0)}_{\text{forandr. i retning } h\in\mathbb{R}^n} + \overbrace{|h|\underbrace{\varepsilon(h)}_{\varepsilon(h)\to 0 \text{ når } |h|\to 0}}^{\text{liten}}$$

Den deriverte til F er ∇F.

Merk: h er en (liten) vektor!

- $\underbrace{F(x_0) + h\cdot\nabla F(x_0)}_{\text{lineær i } h}$ er lineariseringen til F i x_0 i retning h.

- $|h|\varepsilon(h)$ kan skrives $o(h)$ ('lille o'), betyr $\frac{o(h)}{|h|} \to 0$ når $|h|\to 0$.

Eks. $F(x_1, x_2) = x_1^2 + x_2^2$. $\quad DF(x_1, x_2) = [2x_1, 2x_2]$

$$F(x_1 + h_1, x_2 + h_2) = \boxed{(x_1^2 + x_2^2) + \underbrace{2x_1h_1 + 2x_2h_2}_{DF \cdot h}} + o(h)$$

$h = (h_1, h_2)$

Er dette riktig? (er $x_1^2 + x_2^2$ der. bar?)

Sjekk: $(x_1 + h_1)^2 = x_1^2 + 2h_1x_1 + h_1^2$

$$\Rightarrow F(x_1 + h_1, x_2 + h_2) = \boxed{(x_1^2 + x_2^2) + 2h_1x_1 + 2h_2x_2} + h_1^2 + h_2^2$$

$o(h)$ fordi $\dfrac{h_1^2 + h_2^2}{|h|} = \dfrac{|h|^2}{|h|} = \underbrace{|h|}_{\varepsilon(h)} \to 0$ når $|h| \to 0$

F er kont. der. bar dersom de partielle deriverte

$\dfrac{\partial F}{\partial x_j}$ er kont. $\forall j = 1, \dots, n$.

Det vil si: $F(x_0 + h) = F(x_0) + h \cdot \underbrace{DF(x_0)}_{} + |h|\varepsilon(h)$

kont. i x_0!

Eks. $(x_1, x_2) \mapsto x_1^2 + x_2^2$ kont.

der. bar fordi: $x_1 \mapsto 2x_1$, $x_2 \mapsto 2x_2$ er kont.