Bevis omvendte f.-teorem (M. Spivak, 1940–2020)

La $\Lambda = DF[x_0] : \mathbb{R}^n \to \mathbb{R}^n$ med $\det(\Lambda) \neq 0$.

invertibar $n \times n$-matrise.

$$D(\underbrace{\Lambda^{-1} F}_{G})(x_0) = \Lambda^{-1} DF(x_0) = \Lambda \Lambda^{-1} = Id = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

Λ invertibar $\Rightarrow$ dersom $\exists G^{-1}$, så $\exists F^{-1} = G^{-1} \circ \Lambda^{-1}$

og $D(F^{-1}) = D(G^{-1}) \circ \Lambda^{-1}$. $[F \circ F^{-1} = (\Lambda G) \circ (G^{-1} \circ \Lambda^{-1}) = \Lambda \Lambda^{-1} = Id.]$

Så beviser for G med $DG(x_0) = Id : \mathbb{R}^n \to \mathbb{R}^n$.

G C^1: $G(x_0 + h) = G(x_0) + \underbrace{DG[x_0](h)}_{[Id][h]} + |h| \underbrace{\varepsilon(h)}_{\to 0 \text{ då } h \to 0}$

$$\Rightarrow \frac{|G(x_0+h) - G(x_0)|}{|h|} \neq 0 \text{ når } 0 < |h| < \delta.$$

$\uparrow$ $\exists \delta > 0$

$\Rightarrow G(x) \neq G(x_0)$ for x nær x_0 $(x \in B_\delta(x_0))$

G $C^1 \Leftrightarrow DG$ kont $\Rightarrow DG(x) = \overset{DG[x_0] = id}{id} + \underbrace{A(x)}_{\to 0 \text{ då } x \to x_0.}$

Betrakt $|(G(x)-x)-(G(y)-y)|$

$$\underset{\Delta\text{-ulikh.}}{\le} \sum_{j=1}^{n} |(G_j(x)-x_j)-(G_j(y)-y_j)|$$

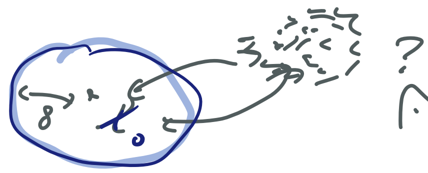
$$\underset{\text{middelv.sats}}{\le} \sum_{j=1}^{n} |\underbrace{\nabla(G_j(x)-x_j)}_{I+A_j(x)}|_{x=c_j}\, |x_j-y_j| \le \underbrace{\sum_{j=1}^{n} |A_j(c_j)|}_{\le \frac{1}{n} \text{ dersom } x,y \text{ nær } x_0 \text{ (velg } \delta \text{ liten)}} |x-y|$$

Men vi har også:

$$\underbrace{|(G(x)-x)-(G(y)-y)|}_{\le \frac{1}{2}|x-y|} \underset{\text{omv. }\Delta\text{-ulikh}}{\ge} |x-y| - |G(x)-G(y)|$$

$$\Rightarrow \boxed{|x-y| \le 2\,|G(x)-G(y)|} \quad \begin{pmatrix} G \text{ injektiv på } B_\delta(x_0) \\ x\neq y \Rightarrow G(x)\neq G(y) \end{pmatrix}$$

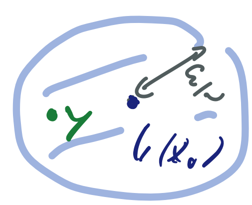
G surjektiv? (dvs, på hva?)



$x \mapsto |G(x)-G(x_0)|$ kont, $\underbrace{\{|x-x_0|=\delta\}}_{\partial B_\delta}$ kompakt

$$\underset{\text{Ekstremalverdi.}}{\Rightarrow} \exists \underbrace{\min_{x\in\partial B_\delta} |G(x)-G(x_0)|}_{\varepsilon} > 0$$

Betrakt $B_{\varepsilon/2}(G(x_0))$



$\boxed{\text{Vil vise: } \forall y \in B_{\varepsilon/2}(G(x_0))\ \exists!\ x \in B_\delta(x_0) : G(x) = y.}$

(unik)

Hvordan? Fikser y og minimerer avstanden

$$h(x) = |y - G(x)|^2 = \sum_{j=1}^{n} (y_j - G_j(x))^2$$

$(y - G(x)) \cdot (y - G(x))$

h kont på $\overline{B_\delta(x_0)} \Rightarrow \exists \min_{\overline{B_\delta(x_0)}} h.$

$x \in \partial B_\delta$?

$$|y - G(x)| \overset{\Delta\text{-ulik.}}{\geq} \underbrace{|G(x) - G(x_0)|}_{\geq \varepsilon} - \underbrace{|G(x_0) - y|}_{< \frac{\varepsilon}{2}} > \frac{\varepsilon}{2} > |y - G(x_0)|$$

Så min ikke på randen, da $|y - G(x_0)| < |y - G(x)|$.

$$\Rightarrow \exists x_y \in B_\delta : \nabla h(x_y) = 0$$

$$2(y - G(x_y)) \cdot \underbrace{DG(x_y)}_{} = 0 \Rightarrow \boxed{y - G(x_y) = 0}$$

$$DG(x) = I + \underset{x \to x_0}{A(x)}, \quad A(x) \to 0$$

Har også: $\boxed{} \Rightarrow G^{-1}$ kont.

Sammen gir dette: G injektiv og surjektiv:

$$\underbrace{G^{-1}\left(B_{\varepsilon/2}(G(x_0))\right)}_{\text{åpen, og inneholder } x_0.} \longrightarrow B_{\varepsilon/2}(G(x_0)).$$

Nå: G^{-1} C^1.

G $C^1 \Rightarrow G(x_1) = G(x) + DG(x)(x_1 - x) + |x_1 - x|\,\varepsilon(x_1 - x)$

$\overset{[DG(x)]^{-1}}{\Longrightarrow} x = x_1 + [DG(x)]^{-1}(G(x_1) - G(x))$

$+ |x_1 - x|\,[DG(x)]^{-1}\varepsilon(x_1 - x)$

$\overset{G \text{ bijekt.}}{\Longleftrightarrow} G^{-1}(y) = G^{-1}(y_1) + [DG(x)]^{-1}(y_1 - y)$

$$\boxed{+|G^{-1}(y_1) - G^{-1}(y)|\,[DG(x)]^{-1}\,\varepsilon(G^{-1}(y_1) - G^{-1}(y))}\ ?$$

Vet $|G^{-1}(y_1) - G^{-1}(y)| \leq 2|y_1 - y| \Rightarrow \boxed{}$

$\leq |y - y_1|\,\tilde{\varepsilon}(y - y_1)$

Så G^{-1} deriverbar i y, med

$$\boxed{D(G^{-1})(y) = [DG(x)]^{-1} = (DG)^{-1} \circ G^{-1}(y)}$$