MA1103 Vector Calculus
Spring 2022

Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 6
Deadline: Feb. 27

You may write solutions in Norwegian or English, as preferable. Note that solutions should be written clearly and with justified arguments.

You can pose questions regarding homework or lecture etc. on the discussion forum Digital Mattelab, see https://wiki.math.ntnu.no/ma1103/2022v/start.

Comment: We mark the source of problems according to the book *Calculus: A Complete Course, 9th edition*, e.g. (12.4) means Chapter 12, Section 4. The source from other editions may be different.

$\boxed{1}$ (12.4)

Let u and v be two twice continuously differentiable real-valued functions defined on an open subset $U \subset \mathbb{R}^2$ which satisfy the so-called *Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y),$$
$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$$

for all $(x,y) \in U$. Show that u and v are *harmonic* in U. That is, satisfy the 2-dimensional Laplace equation $\Delta(\cdot) = 0$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, in U.

Hint: it will be necessary to use and refer to a certain theorem.

$\boxed{2}$ (12.9)

Find the Taylor series for the given functions near the indicated points.

a) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto 2x^2 - xy - y^2 - 6x - 3y + 5, \qquad (x_0, y_0) = (1, -2)$

b) $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto \sin(2x + 3y), \qquad (x_0, y_0) = (0, 0)$

$\boxed{3}$ (13.1)

Find and classify the critical points of the given functions below.

a) $f(x,y) = x^2 + 2y^2 - 4x + 4y$

b) $f(x,y) = x\sin(y)$

$\boxed{4}$ (13.1) *Old exam problem.*

Let $f(x, y) = (x^2 + y^2)e^x$.

a) Find and classify all critical points.

b) Find the tangent plane of the graph $z = f(x, y)$ at the point $(0, 1, 1)$.

$\boxed{5}$ (13.1, 13.2)

a) Find the maximum and minimum values of $f(x, y) = xy - y^2$ on the disk $x^2 + y^2 \leq 1$.

b) Find the maximum and minimum values of $f(x, y) = \sin(x)\cos(y)$ on the closed triangular region bounded by the coordinate axes and the line $x + y = 2\pi$.

How do you know that such extreme values must exist in **a)** and **b)**?

$\boxed{6}$ (12.3, 12.6) *Old exam problem.*

Let

$$f(x, y) := \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that

a) f is continuous at $(0, 0)$,

b) $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

$\boxed{7}$ (inverse function theorem)

Consider the following system of equations

$$u = x\cos(y) \quad \text{and} \quad v = 2x\sin(y).$$

Show that near x_0, y_0 with $x_0 \neq 0$, (x, y) can be expressed as differentiable function of (u, v) and compute $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ near (x_0, y_0).

Hint: check the conditions before using the inverse function theorem.

$\boxed{8}$ (implicit function theorem)

Show that the equations

$$\begin{cases} xy^2 + zu + v^2 = 3, \\ x^3 z + 2y - uv = 2, \\ xu + yv - xyz = 1, \end{cases}$$

can be solved for x, y, and z as functions of u and v near the point P_0 where $(x, y, z, u, v) = (1, 1, 1, 1, 1)$, and find $\frac{\partial y}{\partial u}$ at $(u, v) = (1, 1)$.

Hint: check the conditions before using the implicit function theorem.