Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1103 Vector Calculus
Spring 2022

**Exercise set 13**
**Deadline: May. 1**

You may write solutions in Norwegian or English, as preferable. Note that solutions should be written clearly and with justified arguments.

You can pose questions regarding homework or lecture etc. on the discussion forum Digital Mattelab, see https://wiki.math.ntnu.no/ma1103/2022v/start.

$\boxed{1}$ (16.3)

Compute the area of the region bounded by (på norsk: avgrenset av) the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

using a *line integral.*

$\boxed{2}$ (16.3)

Let $C$ be a simple smooth closed curve in the plane which surrounds the origin, oriented counter-clockwise, and let

$$\mathbf{F}(x,y) = \frac{1}{x^2+y^2}\,(-y,x)\,, \quad (x,y) \neq (0,0).$$

Use Green's Theorem to show that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi.$$

*Hint: We cannot apply Green's Theorem* directly *(!) Why not?*

$\boxed{3}$ (16.4)

Let $U \subset \mathbb{R}^3$ be an open subset containing a body $M \subset \mathbb{R}^3$ whose boundary $\partial M$ is a smooth positively oriented surface. Let $\nu$ denote the unit normal vector field along $\partial M$ and suppose that $u, v$ are smooth real-valued functions defined on $u$. Prove the following so-called *Green's formulae*:

(i)

$$\iiint_M \Delta u\, dV = \iint_{\partial M} \frac{\partial u}{\partial \nu}\, dS.$$

(ii)

$$\iiint_M \nabla u \cdot \nabla v \, dV = \iint_{\partial M} u \frac{\partial v}{\partial \nu} \, dS - \iiint_M u \Delta v.$$

(iii)

$$\iiint_M (u\Delta v - v\Delta u) \, dV = \iint_{\partial M} \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, dS.$$

$\boxed{4}$ *Old exam problem.* (16.4)

Let $\mathbf{F} : (x, y, z) \mapsto (xy^2, y, z)$. Use the Divergence Theorem to compute

$$\iint_T \mathbf{F} \cdot \mathbf{N} \, dS,$$

where $T$ is the part of the sphere $x^2 + y^2 + z^2 = 4$ with $y \geq 0$ and $\mathbf{N}$ is the outward unit normal vector of the sphere.

$\boxed{5}$ (16.5)

Use Stokes's Theorem to show that

$$\int_C y \, dx + z \, dy + x \, dz = \sqrt{3}\pi a^2,$$

where $C$ is the suitably oriented intersection of the surfaces $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$.

$\boxed{6}$ (16.5)

Let $L$ be the curve given by the intersection of the two surfaces $(x - 1)^2 + 4y^2 = 16$ and $2x + y + z = 3$, oriented counterclockwise when viewed from high on the $z$-axis. Let

$$\mathbf{F} = (z^2 + y^2 + \sin x^2)\mathbf{i} + (2xy + z)\mathbf{j} + (xz + 2yz)\mathbf{k}.$$

Evaluate $\displaystyle\int_L \mathbf{F} \cdot d\mathbf{r}$.