MA1103 Solutions for Exercise Set 8

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Problem 1

First, by the extreme-value theorem, we know that f assumes extreme values on D. Critical points:

$$\frac{\partial f}{\partial x} = y = 0$$
$$\frac{\partial f}{\partial y} = x = 0.$$

That is, the origin $(0, 0)$. This is an interior point of D. Next, we need to consider $f|_{\partial D}$ (the restriction of f to ∂D) and look for extreme values here. Consider

$$g(x, y) = x^2 + xy + y^2 - 3.$$

Then $\partial D = \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$. Lagrange's method therefore gives multiplier λ such that

$$\nabla f = \lambda \nabla g.$$

That is,

$$(y, x) = \lambda (2x + y, x + 2y).$$

That is,

$$y = \lambda(2x + y)$$
$$x = \lambda(2y + x).$$

That is,

$$y(1 - \lambda) = \lambda 2x$$
$$x(1 - \lambda) = \lambda 2y.$$

Note first that we may assume $\lambda \neq 0$. Else we end up with with the critical point $(0,0)$ which does not belong to ∂D. Similarly we may assume $\lambda \neq 1$. Thus, we find

$$\begin{aligned} y\frac{1-\lambda}{\lambda} &= 2x \\ x\frac{1-\lambda}{\lambda} &= 2y. \end{aligned}$$

If $x = 0$, we see that $y = 0$ and vice versa. Hence, we may assume $x \neq 0, y \neq 0$. Then dividing we find

$$\frac{2x}{y} = \frac{2y}{x}$$

or

$$2x^2 = 2y^2.$$

That is, $x^2 = y^2$. That is, $x = \pm y$. Substituted into $g(x,y) = 0$ we find $2x^2 \pm x^2 = 3$. If we choose $+$ we get $3x^2 = 3$ or $x^2 = 1$. That is, $x = \pm 1 = y$. If we choose $-$, we get $x^2 = 3$ or $x = \pm\sqrt{3} = -y$. This gives the points $\pm(1,1)$ and $\sqrt{3}(\pm 1, \mp 1)$. That is, four points:

$$p_1 = (1,1), \quad p_2 = (-1,-1), \quad p_3 = \sqrt{3}(1,-1), \quad p_4 = \sqrt{3}(-1,1).$$

Substituted into $f(x,y)$ gives

$$f(0,0) = 0, \quad f(p_1) = 1 = f(p_2), \quad f(p_3) = -3 = f(p_4).$$

Thus the maximum value of f on D is 1 and the minimum value is -3.

Problem 2

$$\begin{cases} x = u^3 + v^3, \\ y = uv - v^2, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} u = u(x,y), \\ v = v(x,y). \end{cases}$$

Differentiating the given equations with respect to x, we get

$$\begin{aligned} 1 &= 3u^2\frac{\partial u}{\partial x} + 3v^2\frac{\partial v}{\partial x}, \\ 0 &= v\frac{\partial u}{\partial x} + (u-2v)\frac{\partial v}{\partial x}. \end{aligned}$$

At $u = v = 1$, we have

$$\begin{aligned} 1 =& 3\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial x}, \\ 0 =& \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{1}{6}.$$

Similarly, differentiating the given equations with respect to y and putting $u = v = 1$, we get

$$\begin{aligned} 0 =& 3\frac{\partial u}{\partial y} + 3\frac{\partial v}{\partial y}, \\ 1 =& \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}. \end{aligned}$$

Thus

$$\frac{\partial u}{\partial y} = \frac{1}{2}, \quad \frac{\partial v}{\partial y} = -\frac{1}{2}.$$

Finally, at $u = v = 1$, we have

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{6}.$$

Problem 3

a) Note that $f(x,y) = e^{g(x,y)}$ where $g(x,y) = -2x^2 - 4xy - y^4$. Thus, the critical points of f are those of g. Computing we find

$$\nabla g(x,y) = \big(-4x - 4y, -4x - 4y^3\big) = -4(x + y, x + y^3).$$

Solving $\nabla g(x,y) = 0$ gives $x = -y$ and $x - x^3 = x(1 - x^2) = 0$. This latter one gives $x^2 = 1$ or $x = 0$. From $x^2 = 1$ we find $x = \pm 1 = -y$. This gives the critical points

$$p_1 = (0,0), \quad p_2 = (1,-1), \quad p_3 = (-1,1).$$

b)

$$\frac{\partial f}{\partial x} = e^g \frac{\partial g}{\partial x} \implies \frac{\partial^2 f}{\partial x^2} = e^g \left(\frac{\partial g}{\partial x}\right)^2 + e^g \frac{\partial^2 g}{\partial x^2} = e^g \left(\left(\frac{\partial g}{\partial x}\right)^2 + \frac{\partial^2 g}{\partial x^2}\right).$$

By symmetry we also find

$$\frac{\partial^2 f}{\partial y^2} = e^g \left(\left(\frac{\partial g}{\partial y}\right)^2 + \frac{\partial^2 g}{\partial y^2}\right),$$

and since f and g are smooth,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = e^g \frac{\partial g}{\partial y}\frac{\partial g}{\partial x} + e^g \frac{\partial^2 g}{\partial y \partial x} = e\left(\frac{\partial g}{\partial y}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial y \partial x}\right).$$

Put $f_{xx} := A, f_{xy} := B, f_{xy} := C$. Then the Hessian of f is given by

$$H(f) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

with determinant $\det(H(f)) = AC - B^2$. Computing we find

$$\begin{aligned} g(x,y) &= -(2x^2 + 4xy + y^4) \\ \frac{\partial g}{\partial x}(x,y) &= -4(x+y) \\ \frac{\partial g}{\partial y}(x,y) &= -4(x+y^3) \\ \frac{\partial^2 g}{\partial x^2}(x,y) &= -4 \\ \frac{\partial^2 g}{\partial y^2}(x,y) &= -12y^2 \\ \frac{\partial^2 g}{\partial y \partial x}(x,y) &= -4 \end{aligned}$$

Now consider each of the points p_1, p_2, p_3 above. We find

- For $p_1 = (0,0)$:

$$\begin{aligned} A(p_1) &= e^{g(0,0)}((g_x(0,0))^2 + g_{xx}(0,0)) = e^0(0^2 + (-4)) = -4 \\ C(p_1) &= e^{g(0,0)}((g_y(0,0)^2 + g_{yy}(0,0)) = e^0(0^2 + 0) = 0 \\ B(p_1) &= -4. \end{aligned}$$

Thus $\det(H(f))(p_1) = -16 < 0$. So $p_1 = (0,0)$ is a saddle point.

- For $p_2 = (1,-1)$:

$$\begin{aligned} A(p_2) &= e^{(1,-1)}((g_x(1,-1))^2 + g_{xx}(1,-1)) = e^{-(3-4)}(0-4) = -4e \\ C(p_2) &= e^{(1,-1)}((g_y(1,-1))^2 + g_{yy}(1,-1)) = e^{-(3-4)}(0-12) = -12e \\ B(p_2) &= -4. \end{aligned}$$

  Thus $\det(H(f))(p_2) = 48e^2 - 16 > 0$, and $A(p_2) < 0$. So $p_2 = (1,-1)$ is a local maximum.

- For $p_3 = (-1,1)$: by symmetry, p_3 is also a local maximum.

c) Let $h(x,y) = x^2 + y^2 - 1$. Then we are looking for the maximum and minimum values of g along $\{(x,y) \in \mathbb{R}^2 : h(x,y) = 0\}$. Lagrange's method gives multiplier λ such that

$$\nabla g = \lambda \nabla h.$$

That is,

$$5\left((x-y)^4, -(x-y)^4\right) = 2\lambda(x,y).$$

That is,

$$\begin{aligned} (x-y)^4 &= \frac{2}{5}\lambda x \\ -(x-y)^4 &= \frac{2}{5}\lambda y. \end{aligned}$$

Suppose that $\lambda \neq 0$. Then, $x = -y$. Substituted into $h(x,y) = 0$ gives $2x^2 = 1$ or $x = \pm\frac{1}{\sqrt{2}} = -y$. This gives the two points $q_1 = \frac{1}{\sqrt{2}}(1,-1)$ and $q_2 = \frac{1}{\sqrt{2}}(-1,1)$. Substituted into the expression for $g(x,y)$ gives

$$\begin{aligned} g(q_1) &= \left(\frac{1}{\sqrt{2}}\right)^5 (1-(-1))^5 = (\sqrt{2})^5 = 4\sqrt{2} \\ g(q_2) &= -4\sqrt{2}. \end{aligned}$$

Now consider $\lambda = 0$. We get $\nabla g(x,y) = 0$ which gives $x = y$. Note that $g|_{\{x=y\}} = 0$. Thus $g(q_1)$ and $g(q_2)$ above are the maximum and minimum values of g along the given curve respectively.

Problem 4

Let $L = x + \lambda(x + y - z) + \mu(x^2 + 2y^2 + 2z^2 - 8)$. For critical points of L:

$$0 = \frac{\partial L}{\partial x} = 1 + \lambda + 2\mu x \tag{1}$$
$$0 = \frac{\partial L}{\partial y} = \lambda + 4\mu y \tag{2}$$
$$0 = \frac{\partial L}{\partial z} = -\lambda + 4\mu z \tag{3}$$
$$0 = \frac{\partial L}{\partial \lambda} = x + y - z \tag{4}$$
$$0 = \frac{\partial L}{\partial \mu} = x^2 + 2y^2 + 2z^2 - 8. \tag{5}$$

From (2) and (3), we have $\mu(y + z) = 0$. Thus $\mu = 0$ or $y + z = 0$.

CASE I. $\mu = 0$. Then $\lambda = 0$ by (2), and $1 = 0$ by (1), so this case is not possible.

CASE II. $y + z = 0$. Then $z = -y$ and, by (4), $x = -2y$. Therefore, by (5), $4y^2 + 2y^2 + 2y^2 = 8$, and so $y = \pm 1$. From this case we obtain two points: $(2, -1, 1)$ and $(-2, 1, -1)$.

The function $f(x, y, z) = x$ has maximum value 2 and minimum value -2 when restricted to the curve $x + y = z$, $x^2 + 2y^2 + 2z^2 = 8$.

Problem 5

$$\int_0^\infty \int_0^\infty e^{-x} y e^{-y^2}\, dx\, dy = \left(\int_0^\infty y e^{-y^2}\, dy\right)\left(\int_0^\infty e^{-x}\, dx\right) = -\frac{1}{2} e^{-y^2} \big|_0^\infty = \frac{1}{2}.$$

Problem 6

a) Using polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4 + x^3 y^3}{x^4 + y^4} = \lim_{r\to 0} \frac{r^4 \cos^4\theta + r^4 \sin^4\theta + r^6 \cos^3\theta \sin^3\theta}{r^4 \cos^4\theta + r^4 \sin^4\theta} = 1.$$

b) Let $f(x, y) = \sin\left(\frac{x^3 y}{x^4 + y^4}\right)$. Then $f(x, x) = \sin\left(\frac{1}{2}\right)$, while $f(0, 1) = 0$. Thus the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Problem 7

The chain rule gives

$$f_x = f_r r_x + f_\theta \theta_x$$
$$f_y = f_r r_y + f_\theta \theta_y.$$

We have $x^2 + y^2 = r^2$. Let $z \in \{x, y\}$. Partial differentiating with respect to z gives $2rr_z = 2z$. That is, $r_z = \frac{z}{r}$. That is, $r_x = \cos\theta, r_y = \sin\theta$. Let $t_x(\theta) = \cos(\theta), t_y = \sin(\theta)$ (the t here stands for *trigonometric* and the subscript z is simply an index and *not* a partial differentiation). Then from $z = rt_z(\theta)$, differentiating with respect to z we find $1 = r_z t_z + r(t_z)'\theta_z$. That is, $\theta_z = \frac{1 - r_z t_z}{r(r_z)'}$. That is,

$$\theta_x = \frac{1 - r_x \cos\theta}{-r\sin\theta} = \frac{1 - \cos^2\theta}{-r\sin\theta} = \frac{\sin\theta}{r}$$
$$\theta_y = \frac{1 - \sin^2\theta}{r\cos\theta} = \frac{\cos\theta}{r}.$$

Note that $\mathbf{u} = (\cos\theta, \sin\theta), \mathbf{u}^\perp = (-\sin\theta, \cos\theta)$ (recall $\mathbf{u}^\perp$ is $\mathbf{u}$ rotated by $\frac{\pi}{2}$ *counter-clockwise*). Thus,

$$\begin{aligned}
\nabla_{(x,y)} f &= (f_r r_x + f_\theta \theta_x, f_r r_y + f_\theta \theta_y) \\
&= \left(f_r \cos\theta - f_\theta \frac{\sin\theta}{r}, f_r \sin\theta + f_\theta \frac{\cos\theta}{r} \right) \\
&= f_r(\cos\theta, \sin\theta) + \frac{1}{r} f_\theta(-\sin\theta, \cos\theta) \\
&= f_r \mathbf{u} + \frac{1}{r} f_\theta \mathbf{u}^\perp.
\end{aligned}$$

Hence we get that $a = f_r, b = \frac{1}{r} f_\theta$.

Problem 8

a) Rewrite the equations into $f_1 = x^2 + 2y^2 + u^2 + v - 6 = 0$ and $f_2 = 2x^3 + 4y^2 + u + v^2 - 9 = 0$. Define $\mathbf{F} : \mathbb{R}^4 \to \mathbb{R}^2$ to be

$$\mathbf{F}(x, y, u, v) = (x^2 + 2y^2 + u^2 + v - 6, 2x^3 + 4y^2 + u + v^2 - 9).$$

Step 1: Check $\mathbf{F}$ is $\mathbf{C^1}$ near $(\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{2})$.
A quick differentiation yields

$$\frac{\partial f_1}{\partial x} = 2x, \quad \frac{\partial f_1}{\partial y} = 4y, \quad \frac{\partial f_1}{\partial u} = 2u, \quad \frac{\partial f_1}{\partial v} = 1, \quad \frac{\partial f_2}{\partial x} = 6x^2, \quad \frac{\partial f_2}{\partial y} = 8y, \quad \frac{\partial f_2}{\partial u} = 1, \quad \frac{\partial f_2}{\partial v} = 2v.$$

Since all partial derivatives exist and continuous everywhere, F is C^1 everywhere (in particular near $(1,-1,-1,2)$).

Step 2: Check $\mathbf{F}(\mathbf{p})=0$.
Clearly $\mathbf{F}(1,-1,-1,2)=(0,0)$.

Step 3: Check $\det\frac{\partial(f_1,f_2)}{\partial(u,v)}(\mathbf{p})\neq 0$**.**

$$\det\frac{\partial(f_1,f_2)}{\partial(u,v)}(\mathbf{p})=\det\begin{pmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{pmatrix}\Bigg|_{(1,-1,-1,2)}=\det\begin{pmatrix}2u & 1\\ 1 & 2v\end{pmatrix}\Bigg|_{(1,-1,-1,2)}=-9\neq 0.$$

Hence, by Implicit Function Theorem, (u,v) can be expressed as a differentiable function of (x,y) near $(1,-1,-1,2)$.

b) To find $\frac{\partial u}{\partial x}$ (denote by u_x) and $\frac{\partial u}{\partial y}$ (denote by u_y), we differentiate both sides of each equation with respect to x (i.e. implicit differentiation), it gives

$$\begin{cases}2x+2uu_x+v_x=0,\\ 6x^2+u_x+2vv_x=0,\end{cases}\implies\begin{cases}2uu_x+v_x=-2x,\\ u_x+2vv_x=-6x^2,\end{cases}\implies\begin{pmatrix}2u & 1\\ 1 & 2v\end{pmatrix}\begin{pmatrix}u_x\\ v_x\end{pmatrix}=\begin{pmatrix}-2x\\ -6x^2\end{pmatrix}.$$

Solving the above system, near $(1,-1,-1,2)$, we have

$$u_x=\frac{6x^2-4xv}{4uv-1}\quad\text{and}\quad v_x=\frac{2x-12x^2u}{4uv-1}.$$

Hence,

$$u_x(1,-1,-1,2)=\frac{2}{9},\quad v_x(1,-1,-1,2)=-\frac{14}{9}.$$