MA1103 Solutions for Exercise Set 5

Norwegian University of Science and Technology

Problem 1

$$
\begin{aligned}
f(x,y) &= \frac{x-y}{x+y}, \quad f(1,1) = 0, \\
f_x(x,y) &= \frac{(x+y)-(x-y)}{(x+y)^2}, \quad f_x(1,1) = \frac{1}{2}, \\
f_y(x,y) &= \frac{(x+y)(-1)-(x-y)}{(x+y)^2}, \quad f_y(1,1) = -\frac{1}{2}.
\end{aligned}
$$

Tangent plane to $z = f(x,y)$ at $(1,1)$ has equation $z = \frac{x-1}{2} - \frac{y-1}{2}$, or $2z = x - y$.
Normal line: $2(x-1) = -2(y-1) = -z$.

Problem 2

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= 2xy + e^{y\cos(xyz)}(-y)\sin(xz)z \\
\frac{\partial f}{\partial y}(x,y,z) &= x^2 + e^{y\cos(xz)}\cos(xz) \\
\frac{\partial f}{\partial z}(x,y,z) &= 1 + e^{y\cos(xz)}(-y)\sin(xz)x,
\end{aligned}
$$

which gives the gradient

$$
\begin{aligned}
\nabla f : (x,y,z) &\mapsto \left(2xy - e^{y\cos(xyz)}yz\sin(xz), x^2 + e^{y\cos(xz)}\cos(xz), 1 - e^{y\cos(xz)}yx\sin(xz)\right) \\
&= (2xy, x^2, 1) - e^{y\cos(xz)}\left(yz\sin(xz), -\cos(xz), yx\sin(xz)\right).
\end{aligned}
$$

Problem 3

(a):

$$f(x,y) = x^2 - y^2, \quad f(2,-1) = 3.$$
$$\nabla f(x,y) = 2x\mathbf{i} - 2y\mathbf{j}, \quad \nabla f(2,-1) = 4\mathbf{i} + 2\mathbf{j}.$$

(b): Tangent plane to $z = f(x,y)$ at $(2,-1)$ has equation $4(x-2)+2(y+1) = z-3$, or $4x+2y-z = 3$.
(c): Tangent line to $f(x,y) = 3$ at $(2,-1)$ has equation $4(x-2)+2(y+1) = 0$, or $2x+y = 3$.

Problem 4

$c = f(2,-4) = e^{4-4} = 1$. Computing, $\frac{\partial f}{\partial x}(x,y) = 2xe^{x^2+y} = 2xf(x,y)$, $\frac{\partial f}{\partial y}(x,y) = e^{x^2+y} = f(x,y)$, so the tangent plane at $(2,-4,1)$ is given by the equation

$$\begin{aligned} z &= c + \frac{\partial f}{\partial x}(2,-4)(x-2) + \frac{\partial f}{\partial y}(2,-4)(y+4) \\ &= 1 + 4(x-2) + (y+4) \\ &= 1 - 8 + 4 + 4x + y. \qquad \text{That is,} \\ z &= 4x + y - 3. \end{aligned}$$

Problem 5

$$f(x,y) = \frac{x}{1+y}, \quad \nabla f(x,y) = \frac{1}{1+y}\mathbf{i} - \frac{x}{(1+y)^2}\mathbf{j},$$
$$\nabla f(0,0) = \mathbf{i}, \quad \mathbf{u} = \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}},$$
$$D_{\mathbf{u}} f(0,0) = \mathbf{i} \bullet \left(\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

Problem 6

The chain rule gives

$$h'(t) = \nabla f(t\mathbf{x}) \cdot \mathbf{x}.$$

Integrating this over $[0,1]$ with respect to t, the fundamental theorem of calculus then gives

$$h(1) - h(0) = \int_0^1 h'(t)\, dt = \int_0^1 \nabla f(t\mathbf{x}) \cdot \mathbf{x}\, dt.$$

Since $h(1) = f(\mathbf{x}), h(0) = f(\mathbf{0})$, we get

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{0}) + \int_0^1 \nabla f(t\mathbf{x}) \cdot \mathbf{x}\, dt \\ &= f(\mathbf{0}) + \int_0^1 \sum_{j=1}^n \frac{\partial f}{\partial x_j}(t\mathbf{x}) x_j\, dt \\ &= f(\mathbf{0}) + \sum_{j=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_j}(t\mathbf{x})\, dt \right) x_j. \end{aligned}$$

Let $g_j(\mathbf{x}) = \int_0^1 \frac{\partial f}{\partial x_j}(t\mathbf{x})\, dt$. We see that each g_j is continuous on $B_\varepsilon(\mathbf{0})$. Moreover we also see that we have $g_j(\mathbf{0}) = \int_0^1 \frac{\partial f}{\partial x_j}(\mathbf{0})\, dt = \frac{\partial f}{\partial x_j}(\mathbf{0})$.

Problem 7

$$f(x,y) = xy, \quad \nabla f(x,y) = y\mathbf{i} + x\mathbf{j}, \quad \nabla f(2,0) = 2\mathbf{j}.$$

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. Thus $u_1^2 + u_2^2 = 1$. We have

$$-1 = D_{\mathbf{u}} f(2,0)\mathbf{u} \bullet \nabla f(2,0) = 2u_2$$

if $u_2 = -\frac{1}{2}$, and therefore $u_1 = \pm\frac{\sqrt{3}}{2}$. At $(2,0)$, f has rate of change -1 in the directions $\pm\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

If $-3 = D_{\mathbf{u}} f(2,0) = 2u_2$, then $u_2 = -\frac{3}{2}$. This is not possible for a unit vector $\mathbf{u}$ since $u_1^2 + u_2^2 = 1$, so there is no direction at $(2,0)$ in which f changes at rate -3.

Problem 8

Let the directional derivative of f at p in the direction $\mathbf{v}$ be denoted by $D_{\mathbf{v}} f(p)$. Then, assuming f is differentiable at p, by the chain rule, we have $D_{\mathbf{v}} f(p) = \nabla f(p) \cdot \mathbf{v}$.

a) $\nabla f(x,y) = (zy\cos(xy), zx\cos(xy), \sin(xy))$, $\nabla f(p) = \nabla f\left(\frac{\pi}{2}, 1, 0\right) = (0,0,1)$, so

$$D_{\mathbf{v}} f(p) = D_{\frac{1}{\sqrt{5}}(2,0,-1)} f(p) = \frac{1}{\sqrt{5}}(2,0,-1) \cdot (0,0,1) = -\frac{1}{\sqrt{5}}.$$

b) We have $\mathbf{v} = \frac{\nabla f(p)}{|\nabla f(p)|}$ so $D_{\mathbf{v}} f(p) = |\nabla f(p)|$. A direct computation gives the following expressions for the gradient of f: $\nabla f(x, y) = (-4x, -6y) = -2(2x, 3y), \nabla f(p) = \nabla f(2, 3) = -2(4, 9)$, so

$$D_{\mathbf{v}} f(p) = |\nabla f(p)| = |\nabla f(2, 3)| = 2\sqrt{4^2 + 9^2} = 2\sqrt{16 + 81} = 2\sqrt{97}.$$

Problem 9

Assume $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ and $\mathbf{h} = (h_1, h_2, \cdots, h_n)$. If f is differentiable at $\mathbf{a}$, then

$$\frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - \nabla f(\mathbf{a}) \bullet \mathbf{h}}{|\mathbf{h}|}$$

approaches 0 as $\mathbf{h} \to \vec{0}$. Since the denominator of this fraction approaches zero, the numerator must also approach 0 or the fraction would not have a limit. Since the term $\nabla f(\mathbf{a}) \bullet \mathbf{h}$ approaches 0, we must have

$$\lim_{\mathbf{h} \to \vec{0}} \left[f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) \right] = 0.$$

Thus f is continuous at $\mathbf{a}$.

Problem 10

By uniqueness of the derivative, it suffices to prove that $\lim_{h \to 0} L(h) = 0$ where

$$L(h) = \frac{|(fg)(a + h) - (fg)(a) - (f(a)g'(a) + g(a)f'(a))(h)|}{|h|}.$$

Considering the numerator, adding 0 inside the absolute value, and using the triangle inequality, we find

$$\begin{aligned}
&|fg(a + h) - (fg)(a) - (f(a)g'(a) + g(a)f'(a))(h)| \\
&= |f(a + h)g(a + h) - f(a)g(a) - f(a)g'(a)(h) + g(a)f'(a)(h)| \\
&= |f(a + h)g(a + h) - f(a)g(a) - f(a)g'(a)(h) + g(a)f'(a)(h) + \overbrace{f(a)g(a + h) - f(a)g(a + h)}^{=0}| \\
&= |\left[g(a + h) - g(a) - g'(a)(h)\right] f(a) + \left[f(a + h) - f(a) - f'(a)(h)\right] g(a + h)| \\
&\leq |g(a + h) - g(a) - g'(a)(h)||f(a)| + |f(a + h) - f(a) - f'(a)(h)||g(a + h)|
\end{aligned}$$

Thus,

$$L(h) \leq \frac{|g(a + h) - g(a) - g'(a)(h)|}{|h|} \cdot |f(a)| + \frac{|f(a + h) - f(a) - f'(a)(h)|}{|h|} \cdot |g(a + h)|.$$

Using the definition of $g'(a)$ and $f'(a)$, and noting that $\lim_{h\to 0} |g(a+h)| = |g(a)|$ since g is continuous at a being differentiable at a (Problem 9), letting $h \to 0$ we find

$$\lim_{h\to 0} L(h) \leq 0 \cdot |f(a)| + 0 \cdot |g(a)| = 0.$$

Since the left-hand side is always non-negative, this shows that it is 0, which proves the proposition.