Bevis omvendte f.teor. (M. Spivak, 1940–)

La $\Lambda = DF[x_0] : \mathbb{R}^n \to \mathbb{R}^n$ med $\det(\Lambda) \neq 0$.
inv.bar $n\times n$-matrise.

$$D(\underbrace{\Lambda^{-1}F}_{G})(x_0) = \Lambda^{-1} DF(x_0) = \Lambda^{-1}\Lambda = Id$$

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

Λ invertibar $\Longrightarrow$ dersom $\exists\, G^{-1}$, så $\exists\, F^{-1} = G^{-1} \circ \Lambda^{-1}$
og $D(F^{-1}) = D(G^{-1}) \circ \Lambda^{-1}$.

$$F \circ F^{-1} = (\Lambda \underbrace{G)(G^{-1}}_{id}\Lambda^{-1}) = \Lambda\Lambda^{-1} = Id.$$

Så: vi beviser setn. for G med $DG[x_0] = id : \mathbb{R}^n \to \mathbb{R}^n$.

G C^1: $G(x_0+h) = G(x_0) + \underbrace{DG(x_0)h}_{[Id][h]} + |h|\varepsilon(h)$, $\varepsilon(h) \to 0$ då $h \to 0$

$$\Longrightarrow \frac{|G(x_0+h) - G(x_0)|}{|h|} \neq 0 \quad \text{når } 0<|h|<\delta$$

$\exists\, \delta > 0$

$\Longrightarrow$ $G(x) \neq G(x_0)$ for x nært x_0 $(x \in B_\delta(x_0))$

G $C^1 \iff DG$ kont. $\Longrightarrow$ $DG(x) = id + A(x)$, $DG(x_0) = id$, $A(x) \to 0$ då $x \to x_0$

Nå: viser at G er injektiv på $B_\delta(x_0)$.

Betrakt $\big|(G(x)-x)-(G(y)-y)\big|$

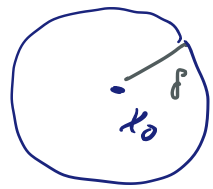
$$\underset{\Delta\text{-ulikh.}}{\le} \sum_{j=1}^{n} \big|(G_j(x)-x_j)-(G_j(y)-y_j)\big|$$

$$\underset{\text{middelv.setn.}}{\le} \sum_{j=1}^{n} \big|\nabla(G_j(x)-x_j)\big|_{x=c}\big|\,|x_j-y_j|$$

$\nabla G_j(x) = 1 + A_j(x)$, $\nabla x_j = 1$ (i $x=c$)

$$\le \sum_{j=1}^{n} |A_j(c_j)|\,|x-y|$$

$\sum_{j=1}^{n} |A_j(c_j)| \le \frac{1}{2}$ dersom x, y nært x_0 (velg δ liten)



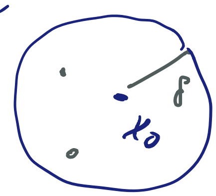
Men har også: $\big|(G(x)-x)-(G(y)-y)\big|$

$$\underset{\Delta\text{-ulikh.}}{\ge} |x-y| - |G(x)-G(y)|$$

$$\Longrightarrow \quad |x-y| \le 2\,|G(x)-G(y)|$$

G injektiv på $B_\delta(x_0)$: $x \ne y \Rightarrow G(x) \ne G(y)$

G surjektiv? (dvs. på hva?)



G !

$x \mapsto |G(x) - G(x_0)|$ kont, $\{|x - x_0| = \delta\}$ komp. ($\bigcirc\, \partial B$)

Bolz.-W.-setn.

$$\Rightarrow \quad \exists \underbrace{\min_{x \in \partial B_\delta(x_0)} |G(x) - G(x_0)|}_{\varepsilon} > 0.$$

Betrakt $B_{\varepsilon/2}(G(x_0))$

Vil vise: $\forall y \in B_{\varepsilon/2}(G(x_0))\ \exists!\ x \in B_\delta(x_0)$: $G(x) = y$.
(! = unik)

Hvordan? Fiksér y og minimér avstanden

$$h(x) = |y - G(x)|^2$$

Har vist: $DG[x_0] = \mathrm{id} \quad \begin{bmatrix} 1 & & 0 \\ & 1 \ddots & \\ 0 & & 1 \end{bmatrix}$

$$|x - y| \leq 2\,|G(x) - G(y)|$$

på 

$$\exists \underbrace{\min_{x \in \partial B_\delta(x_0)} |G(x) - G(x_0)|}_{\varepsilon} > 0.$$

Fiks y og minimér avstanden

$$h(x) = |y - G(x)|^2 = \sum_{j=1}^{n} (y_j - G_j(x))^2$$

h kont. på $\overline{B_\delta(x_0)} \Longrightarrow \exists \min_{\overline{B_\delta(x_0)}} h$.

$x \in \partial B_\delta$?

$$|y - G(x)| \overset{\Delta\text{-ulikh.}}{\geq} \underbrace{|G(x) - G(x_0)|}_{\geq \varepsilon} - \underbrace{|G(x_0) - y|}_{< \varepsilon/2}$$

$$> \frac{\varepsilon}{2} > |y - G(x_0)|$$

Så: min h ikke på randen til $B_\delta(x_0)$

$$\Longrightarrow \exists x_y \in B_\delta : \nabla h(x_y) = 0.$$

$$h(x) = (G(x) - y) \cdot (G(x) - y)$$

$$\nabla h(x_y) = 2(G(x_y) - y) \cdot DG(x_y) = 0 \Longrightarrow G(x_y) = y$$

$$DG(x) = Id + A(x), \quad A(x) \xrightarrow{x \to x_0} 0$$

$\Longrightarrow G$ injektiv og surjektiv: $B_\delta(x_0) \to B_{\varepsilon/2}(G(x_0))$

[ ] $\Longrightarrow$ G^{-1} kont.

Nå: G^{-1} C^1

G C^1: $G(x_1) = G(x) + DG[x](x_1 - x) + |x_1 - x|\,\varepsilon(x_1 - x)$

$[DG(x)]^{-1}$ $\Longrightarrow$
$$x = x_1 + [DG(x)]^{-1}(G(x) - G(x_1)) + |x_1 - x|\,[DG(x)]^{-1}\varepsilon(x_1 - x)$$

G bijektiv

$\Longleftrightarrow$
$$G^{-1}(y) = G^{-1}(y_1) + (DG(x))^{-1}(y - y_1) + |G^{-1}(y_1) - G^{-1}(y)|\,(DG(x))^{-1}\varepsilon(G^{-1}(y_1) - G^{-1}(y))$$

[ ] $\Longrightarrow$ $\leq |y - y_1|\,\tilde{\varepsilon}(y - y_1)$

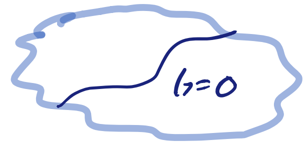
Så G^{-1} derivabar i y, med

$$D(G^{-1})(y) = (DG(x))^{-1} = (DG)^{-1} \circ G^{-1}(y)$$

13.3 Minimering ved bivilkår / Lagrange multiplikatorer

La $F : U \subset \mathbb{R}^n \to \mathbb{R}$, og $G : U \subset \mathbb{R}^n \to \mathbb{R}$.

Hva gjelder for min/maks av F over $\{x \in U : G(x) = 0\}$



Spørsmål 1: Har $\{G(x) = 0\}$ struktur?

IFT: $|\nabla G| \neq 0$ langs $G(x) = 0 \implies$

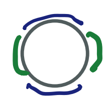
$\exists$ C^1-funksjon slik at $G(x) = 0$ er en kurve / flate / $(n-1)$-dimensjonal hyperflate.

F.eks. i $\mathbb{R}^2$: $G(x_1, x_2)$ med $\frac{\partial G}{\partial x_1} \neq 0$

lokalt $\implies$ $G(x_1, x_2) = 0 \iff G(\phi(s), s) = 0$ kurve!

Eks. $\underbrace{x^2 + y^2}_{G} = 1$, $|\nabla G| = |(2x, 2y)| = (4x^2 + 4y^2)^{\frac{1}{2}}$
$= 2 \neq 0$

$\implies$ Enten $x = x(y)$ eller $y = y(x)$

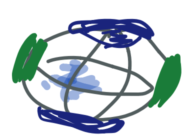


i $\mathbb{R}^3$: $G(x_1, x_2, x_3)$ med $\frac{\partial G}{\partial x_1} \neq 0 \overset{\text{lok.}}{\implies}$

$G(x_1, x_2, x_3) = 0 \iff G(\phi(s,t), s, t) = 0$ flate!

Eks. $x^2 + y^2 + z^2 = 1$, $|\nabla G| = |(\underset{\neq 0}{2x}, \underset{\neq 0}{2y}, 2z)| = 2 \neq 0$

$\implies x = x(y,z)$, $y = y(x,z)$, $z = z(x,y)$.



La γ være en C^1-kurve på $\{G(x)=0\}$, dersom $\gamma : I \to U \subset \mathbb{R}^n$. La $\gamma(0) = x_0$.

Dersom F har et lokalt maks/min i x_0 over $\{G(x)=0\}$ må $\frac{d}{dt} F\circ\gamma(0) = 0 \iff$ (kj.r.)

$$\underbrace{DF(\gamma(0))}_{x_0}\cdot\dot\gamma(0) = 0 \iff DF \perp \dot\gamma$$ for hver kurve γ over $\{G(x)=0\}$.

(tangentvektor!)

$\{G(x)=0\}$ $(n-1)$-dimensjonal

$\Rightarrow$ DF normal til $\{G=0\}$.

Men vet også $G(\gamma)=0 \Rightarrow DG(\gamma)\cdot\dot\gamma = 0$ $\forall \gamma$ på $G=0$.

$\Rightarrow$ DG normal til $\{G=0\}$.

Så $\boxed{DF \,/\!/\, DG \text{ i } x_0}$, dvs

$\exists \lambda \in \mathbb{R}$: $\quad \nabla F(x_0) = \lambda \nabla G(x_0)$

(Lagrangemult.)

eller $DG(x_0) = 0$

Setning (Lagrangemetode.) Et lokalt maks/min for $F \in C^1(U, \mathbb{R})$ over $\{x \in U : G(x) = 0\}$, $G \in C^1(U, \mathbb{R})$ med $\nabla G \neq 0$ må skje i et punkt der $F - \lambda G$ har et kritisk punkt,

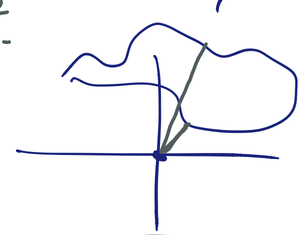
$$\boxed{\nabla F = \lambda \nabla G}, \quad \lambda \in \mathbb{R}.$$

Men setningen sier ingenting om punkter der $\nabla G = 0$, så må sjekke disse også.

Eks. Oppgave 5 8.6 2011

Finn største/minste avstand fra origo til kurven $5x^2 + 6xy + 5y^2 = 8$.

Løsn. Minimér/maksimér $F(x,y) = x^2 + y^2$ over $G(x,y) = 5x^2 + 6xy + 5y^2 = 8$.



$F, G \in C^1 \Rightarrow$ nødvendig vilkår

$\nabla F \parallel \nabla G$, dvs

$\nabla F = \lambda \nabla G$ eller $\nabla G = (0,0)$;

$r = \sqrt{x^2 + y^2}$

$r^2 = x^2 + y^2$

min/maks.

$$\nabla f(x,y) = (2x, 2y) = \lambda(10x+6y,\ 10y+6x)$$
$$= \lambda \nabla g(x,y).$$

$\lambda = 0 \Rightarrow (x,y) = (0,0)$ umulig (ligger ikke på $g(x,y)=8$).

$\nabla g = (0,0) \Leftrightarrow \begin{cases} 5x = -3y \\ 5y = -3x \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ umulig

$$\begin{cases} 2x = 10\lambda x + 6\lambda y & \cdot y \\ 2y = 10\lambda y + 6\lambda x & \cdot(-x) \end{cases}$$

[obs. $y=0 \Rightarrow x=0$, $x=0 \Rightarrow y=0$ ikke på kurven]

$$0 = 0 + 6\lambda(y^2 - x^2) \underset{\lambda\neq 0}{\Leftarrow} x^2 = y^2 \Leftrightarrow \boxed{x = \pm y}$$

Sett inn: $g(x,x) = 16x^2 = 8 \Leftrightarrow \boxed{x = \pm \frac{1}{\sqrt{2}}}$

$g(x,-x) = 4x^2 = 8 \Leftrightarrow \boxed{x = \pm\sqrt{2}}$

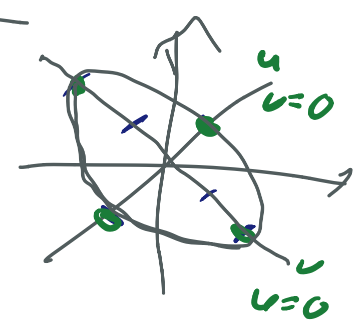
$f(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2} = 1$ lok min f Avstand 1

$f(\pm\sqrt{2}, \mp\sqrt{2}) = 2 + 2 = 4$ lok maks f Avstand 2

Men finnes (globalt) maks/min?

Må være $\{g=8\}$ kompakt, f kont.

Plottet: maks for $x=y$, $r=2$.
min for $x=-y$, $r=1$



$$x = \frac{1}{\sqrt{2}}(u+v) \qquad\qquad u = \frac{1}{\sqrt{2}}(x+y)$$

$$y = \frac{1}{\sqrt{2}}(u-v) \iff v = \frac{1}{\sqrt{2}}(x-y)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}$$

inv. kar med $|DF| = 1$

$(DF)^{-1} = (DF)^t$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

ON-transf.

rotasjon $-\frac{\pi}{4}$

$$5x^2 = \frac{5}{2}(u^2 + \cancel{2uv} + v^2)$$

$$5y^2 = \frac{5}{2}(u^2 - \cancel{2uv} + v^2)$$

$$6xy = \frac{6}{2}(u^2 - v^2)$$

$$\boxed{8u^2 + 2v^2 = 8}$$

$$\implies \boxed{u^2 + \left(\frac{v}{2}\right)^2 = 1}$$

ellipse med

halvakse 1 i retning u

halvakse 2 i retning v
