$\partial_{x_1} F$ og $\partial_{x_2} F$ kont. $\Longrightarrow$

$$\partial_{x_1} F(x_1+h_1, x_2) = \partial_{x_1} F(x_1, x_2) + \underbrace{\varepsilon_1(h)}_{\to 0 \text{ da } h \to 0}$$

$$\partial_{x_2} F(x_1+h_1, x_2+h_2) = \partial_{x_2} F(x_1, x_2) + \underbrace{\varepsilon_2(h)}_{\to 0 \text{ da } h \to 0}$$

$$\Longrightarrow \boxed{F(x+h) = F(x) + h \cdot \nabla F(x) + \underline{h_1 \varepsilon_1(h) + h_2 \varepsilon_2(h)}}$$

La $\varepsilon(h) = \dfrac{h_1 \varepsilon_1(h)}{|h|} + \dfrac{h_2 \varepsilon_2(h)}{|h|} \to 0$ da $|h| \to 0$.

$$\Longrightarrow \quad F(x+h) = F(x) + h \cdot \nabla F(x) + |h| \varepsilon(h).$$

Så F der. bar.

$\square$

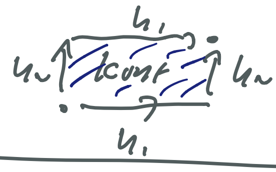
12.4 Høyere deriverte

$\partial_{x_1} \partial_{x_2} F$ presis som man tror.

Eks. $\partial_{x_1} \partial_{x_2} (x_1^2 e^{2x_2}) = \partial_{x_1} (2 x_1^2 e^{2x_2}) = 4 x_1 e^{2x_2}$.

Teorem $\partial_{x_1} \partial_{x_2} F = \partial_{x_2} \partial_{x_1} F$

dersom $\partial_{x_1}^2 F$, $\partial_{x_1} \partial_{x_2} F$, $\partial_{x_2} \partial_{x_1} F$, $\partial_{x_2}^2 F$ er kont.

Uten bevis. Idé 

To 'moteksempler'

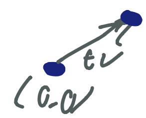
(i) $\exists\, \partial_v F$ men $F \notin C^1$.

(ii) $\exists\, \partial_{x_1}\partial_{x_2} F \neq \partial_{x_2}\partial_{x_1} F$

(i) $\boxed{F : \mathbb{R}^2 \to \mathbb{R},}$ $\quad (x,y) \longmapsto \begin{cases} \dfrac{xy}{(x^2+y^2)^{\frac{1}{2}}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Merk: $\underset{r\neq 0}{F(x,y)} = \dfrac{r^2 \cos\theta \sin\theta}{r} = \dfrac{r}{2}\sin(2\theta) \xrightarrow[r\to 0]{} 0$

F kont på hele $\mathbb{R}^2$.



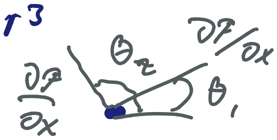
La $v = (\cos\alpha, \sin\alpha)$, $|v| = 1$. Fiksér α.

$$\partial_v F(0,0) \overset{\text{def.}}{=} \lim_{t\to 0} \frac{F(tv) - F(0,0)}{t} = \frac{\frac{t}{2}\sin(2\alpha) - 0}{t} = \frac{1}{2}\sin(2\alpha)$$

$$\Longrightarrow \boxed{\exists\, \partial_v F(0,0) \quad \forall \alpha \in [0, 2\pi)}$$

Men: $\dfrac{\partial F}{\partial x}(x,y) \underset{(x,y)\neq(0,0)}{=} \dfrac{y}{(x^2+y^2)^{\frac{1}{2}}} - \dfrac{\cancel{2}x^2 y}{\cancel{2}(x^2+y^2)^{\frac{3}{2}}} = \underline{\sin\theta(1-\cos^2\theta)}$

og $\dfrac{\partial F}{\partial y}(x,y) = \underline{\cos\theta(1-\sin^2\theta)}$



$\Rightarrow$ DF kont på $\mathbb{R}^2 \setminus \{(0,0)\}$, men savner grenseverdi i $(0,0)$.

$\Rightarrow$ $f \notin C^1(\mathbb{R}^2, \mathbb{R})$

(+ kan vise: f ikke der. bar. i origo.)

(ii) $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{\underbrace{x^2+y^2}_{r^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0). \end{cases}$$

- $f(x,y) = r^2 \underline{\cos\theta \sin\theta}\, (\underline{\cos^2\theta - \sin^2\theta})$

$$= \frac{r^2}{2} \sin(2\theta) \cos(2\theta) = \underbrace{\boxed{\frac{r^2}{4} \sin(4\theta)}}_{\text{kont. på } \mathbb{R}^2}. \xrightarrow{r \to 0} 0$$

- også C^1 (sjekk).

Men $\partial_y(\partial_x f)(0,0) = \cdots = \partial_y(-y) = -1$.

$\partial_x(\partial_y f)(0,0) = \cdots = \partial x(x) = 1.$ $\neq$

Årsak: 'To deriverte i origo eliminerer r^2, og θ blir viktig.'

Nå: funksjoner $D \subset \mathbb{R}^n \to \mathbb{R}^m$.

La oss begynne med $\mathbb{R}^2 \to \mathbb{R}^2$.

Eks. $F : [0,\infty) \times [0, 2\pi) \longrightarrow \mathbb{R}^2$

$$(r,\theta) \longmapsto F(r,\theta) = (\underbrace{x(r,\theta)}_{F_1}, \underbrace{y(r,\theta)}_{F_2})$$

$$= (r\cos\theta, r\sin\theta)$$

En funksjon!

$$x = r\cos\theta$$
$$y = r\sin\theta$$

- Er den kont.?

Ja, fordi komponentene $F_1 : \mathbb{R}^2 \to \mathbb{R}$, $F_2 : \mathbb{R}^2 \to \mathbb{R}$ er kont.

- Er den part. der. bar?

Ja, fordi $\frac{\partial F}{\partial r} = \left(\frac{\partial F_1}{\partial r}, \frac{\partial F_2}{\partial r}\right) = (\cos\theta, \sin\theta)$

og $\frac{\partial F}{\partial \theta} = \left(\frac{\partial F_1}{\partial \theta}, \frac{\partial F_2}{\partial \theta}\right) = (-r\sin\theta, r\cos\theta)$.

- Er F kont. der. bar?

Ja, fordi $\frac{\partial F}{\partial r}$, $\frac{\partial F}{\partial \theta}$ er kont. i (r,θ).

- Hva er tilsvarigheten til DF?

Skriver $\boxed{DF}$ eller J: $\left[\left[\frac{\partial F}{\partial r}\right]\left[\frac{\partial F}{\partial \theta}\right]\right] = \begin{bmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \theta} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \theta} \end{bmatrix}$

'Jacobian

en matrise!

Generelt for $F: \mathbb{R}^n \to \mathbb{R}^m$, er

$$DF = \begin{array}{cc} m & \begin{bmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_n \\ \vdots & \ddots & \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_n \end{bmatrix} \\ & n \end{array}$$

I vårt tilfelle: $DF(r, \theta) = \boxed{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}$

$\det DF(r,\theta) = r\cos^2\theta - (-r\sin^2\theta) = r \neq 0$ utenom i origo. Husk dette, kobler til $(r, \theta) \mapsto (x,y)$ ikke inverterbar i origo.

Hva er bra med Jacobi-matrisen?

Kjerneregel $F: \mathbb{R}^n \to \mathbb{R}^m$, $g: \mathbb{R} \to \mathbb{R}^n$.

$F \circ g : \mathbb{R} \to \mathbb{R}^m$. Tenk på F som m kopier $F_j : \mathbb{R}^n \to \mathbb{R}$. $F = (F_1, \ldots, F_m)$.

$(F_j \circ g)' = \nabla F_j(g) \cdot g' = [\nabla F_j(g)] \begin{bmatrix} g' \end{bmatrix}$.

$$(F\circ g)' = \big[DF(g)\big]\big[g'\big] = \begin{array}{c c c} & & n \\ m & \begin{bmatrix} DF_1(g) \\ \vdots \\ DF_m(g) \end{bmatrix} & \begin{bmatrix} g_1' \\ \vdots \\ g_n' \end{bmatrix} \\ & n & 1 \end{array} \quad \boxed{m\times 1}$$

Hvis nå $G = (g_1(t_1,\dots,t_r), \dots, g_n(t_1,\dots,t_r))$

$\mathbb{R}^r \to \mathbb{R}^n$

blir $\dfrac{\partial(F\circ G)}{\partial t_i} = \big[DF(G)\big]\left[\dfrac{\partial G}{\partial t_i}\right]$ Helt likt!

$t = t_i$ ovenfor

og hele $D(F\circ G) = \begin{array}{c c c c} m & \big[DF(G)\big] & \big[DG\big] & n \\ & n & r & \end{array} \quad \boxed{m\times r}$

blir r kopier

Tilbake til vårt eksempel:

Dersom vi har en funksjon $F(x,y)$ og ønsker bytte til polarkoord.,

$$\boxed{\tilde{F}(r,\theta) = F(x(r,\theta), y(r,\theta)),}$$

Hva er $D\tilde{F}(r,\theta)$ dersom vi kjenner $DF(x,y)$?

La $G(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$

$\mathbb{R}^2 \to \mathbb{R}^2$

$$\Rightarrow \boxed{\tilde{F} = F \circ G.}$$

$$\nabla \tilde{F} = [\nabla F \circ G]\underbrace{[DG]}_{\begin{bmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix}}$$

F. ehs $\quad F(x,y) = xe^y \qquad \mathbb{R}^2 \to \mathbb{R}$

$$\nabla F(x,y) = (e^y, xe^y) = (1, x)e^y.$$

$$\nabla F \circ G(r,\theta) = (1, r\cos\theta)\, e^{r\sin\theta}.$$

$$[\nabla F \circ G]\underbrace{\begin{bmatrix} & DG & \end{bmatrix}}_{\begin{bmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{bmatrix}} = e^{r\sin\theta}\begin{bmatrix}\cos\theta + r\cos\theta\sin\theta \\ -r\sin\theta + r^2\cos^2\theta\end{bmatrix}^t$$

Sickh: $\boxed{r\cos\theta\, e^{r\sin\theta}}$

$$\partial_r \rightsquigarrow (\cos\theta + r\cos\theta\sin\theta)\, e^{r\sin\theta}$$

$$\partial_\theta \rightsquigarrow (-r\sin\theta + r^2\cos^2\theta)\, e^{r\sin\theta}$$
