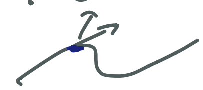
11.4 Krumning, torsjon og Frenetrammen

Krumning forholder seg til fart, (omtrent) som den andrederiverte til den førstederiverte.

Husk: $\gamma : I \to \mathbb{R}^n$, glatt, $\leftarrow \exists\, \dot\gamma(t) \neq 0$



$$T(t) = \frac{v(t)}{|v(t)|} = \frac{\dot\gamma(t)}{|\dot\gamma(t)|} \quad \text{enhetstangent}$$

Def. Krumningen $\kappa \overset{\text{def.}}{=} \left|\frac{dT}{ds}\right|$

↑ buelengdep.

Hva er $\frac{dT}{ds}$?

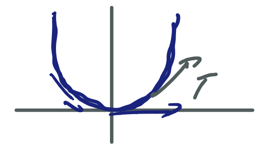
Den deriverte av tangenten når hastigheten er konst 1 langs kurven.

Merk: $\frac{ds}{dt} = |\dot\gamma(t)| = |v(t)| \implies \boxed{\frac{dt}{ds} = \frac{1}{|v(t)|}}$

$$\boxed{\text{Så } \kappa = \left|\frac{dT}{ds}\right| \underset{\text{kj.r.}}{=} \left|\frac{dT}{dt}\frac{dt}{ds}\right| = \frac{1}{|v(t)|}\left|\frac{dT}{dt}\right|}$$

Eks. (i $\mathbb{R}^2$):

(i) parabel $\gamma(t) = (t, t^2)$, $t \in \mathbb{R}$



$$\dot\gamma(t) = v(t) = (1, 2t), \quad |\dot\gamma(t)| = \boxed{|v(t)| = \sqrt{1+4t^2}} \neq 0$$

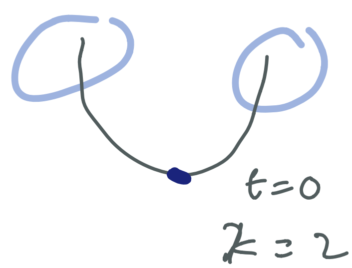
$$T = \frac{v}{|v|} = \frac{(1,2t)}{(1+4t^2)^{\frac{1}{2}}}$$

$$\left|\frac{dT}{dt}\right| \underset{\uparrow\ \text{MA1102}}{=} \left|\frac{(-4t,\ 2)}{(1+4t^2)^{\frac{3}{2}}}\right| = \frac{(4(4t^2)+4)^{\frac{1}{2}}}{(1+4t^2)^{\frac{3}{2}}}$$

$$= \boxed{\frac{2}{1+4t^2}}$$

Så $\kappa = \left|\frac{dT}{dt}\right| \frac{1}{|v|} = \frac{2}{1+4t^2} \cdot \frac{1}{(1+4t^2)^{\frac{1}{2}}} = \boxed{\frac{2}{(1+4t^2)^{\frac{3}{2}}}}$

$\kappa \to 0$ når $t \to \pm\infty$.

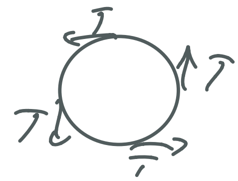


(ii) sirkel $\gamma(\theta) = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$.

$$|\dot\gamma(\theta)| = |(-\sin\theta, \cos\theta)| = 1 \quad \forall\theta$$

Så $\theta = s$ er buelengden.

$$\Rightarrow \quad T = \frac{v}{|v|} = v = (-\sin\theta, \cos\theta)$$



$$\text{og} \quad \kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{d\theta}\right| = \left|\frac{dv}{d\theta}\right| = |(-\cos\theta, -\sin\theta)| = 1.$$

Enhetssirkelen har konstant krumning 1.

Enhetsnormalen

Ettersom $|T(t)| = 1 \quad \forall t$, følger

$T \cdot T' = 0$, dvs T' normal mot T.

Def. Enhetsnormalen

$$\boxed{N \overset{\text{def.}}{=} \frac{dT/ds}{|dT/ds|} = \frac{1}{\kappa}\frac{dT}{ds}.}$$

$$\frac{ds}{dt} = |v| > 0$$

Ved kjerneregelen:

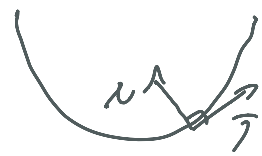
$$\boxed{N = \frac{\frac{dT}{dt}\cdot\cancel{\frac{dt}{ds}}}{\left|\frac{dT}{dt}\cdot\cancel{\frac{dt}{ds}}\right|} = \frac{dT/dt}{|dT/dt|}.}$$

κ er til N, som $|v|$ er til T.

Eks. (i) parabelen $T = \frac{(1,2t)}{\sqrt{1+4t^2}}$ og

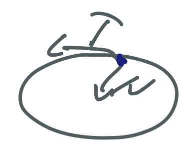
$(-4t,2)$

$N = \frac{(-2t,1)}{\sqrt{1+4t^2}}$ med lengde 1.



$$T\cdot N = \frac{-2t+2t}{1+4t^2} = 0.$$

(ii) sirkelen $\boxed{\begin{aligned} T &= (-\sin\theta, \cos\theta) \\ N &= (-\cos\theta, -\sin\theta) \end{aligned}}$



$T \perp N$

Så i $\mathbb{R}^2$ er $\{T, N\}$ et lokalt (langs kurven) koordinatsystem (ortogonal bas).

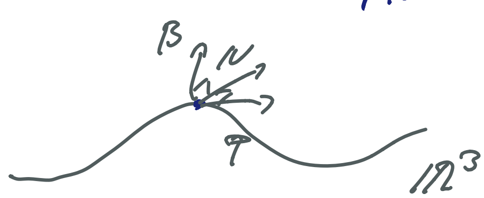
I $\mathbb{R}^3$ kan vi legge til binormalen

$\boxed{B \overset{\text{def.}}{=} T\times N}$

$B \perp T$, $B \perp N$

$|B| = |T\times N| = 1.$

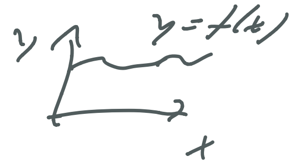
$\boxed{\text{torsjon} \quad \frac{dB}{ds} = -\tau N}$



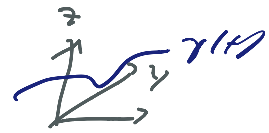
Frenetrammen $\{T, N, B\}$ 3-dim lokalt koord.syst. langs kurven.

12.1 Funksjoner av flere variabler

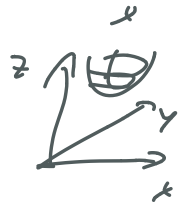
Reelle funksjoner: $f: \mathbb{R} \to \mathbb{R}$



Kurver: $\gamma: \mathbb{R} \to \mathbb{R}^n$



Nå: $F: \mathbb{R}^n \to \mathbb{R}$



$z = F(x,y)$

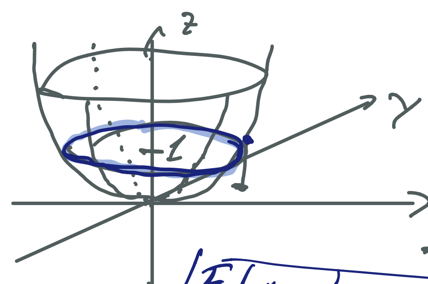
Enkleste eksemplet:

$$\boxed{\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{R} \\ (x,y) &\mapsto x^2+y^2 \end{aligned}}$$

Graf i $\mathbb{R}^3$:

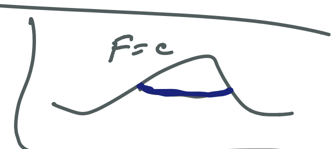
$\{(x,y,z) : z = x^2+y^2\}$

Paraboloide (se 10.5)



$\boxed{F(x,y) = x^2+y^2}$

Grafen er en **overflate**.



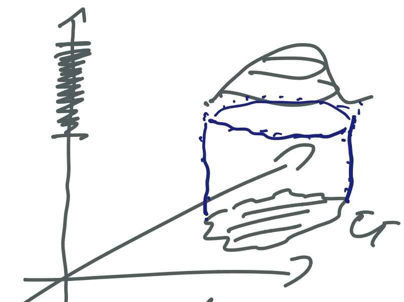
Kurvene $\{F(x,y) = c\}$ er **nivåkurver**.

Generelt $\{F(x_1, x_2, \ldots, x_n) = c\}$ **nivåmengde**.

En funksjon $F: U \subset \mathbb{R}^n \longrightarrow F(U) \subset \mathbb{R}$

$$x = (x_1, x_2, \dots, x_n) \longmapsto F(x) = F(x_1, \dots, x_n)$$

har:



(i) en <u>definisjonsmengde</u> U.

(også domene)

(ii) kodomene $\mathbb{R}$

(iii) Et bilde $\{F(x) : x \in U\} \subset \mathbb{R}$.

Dersom U ikke er gitt, er $\{x \in \mathbb{R}^n : F(x) \in \mathbb{R}\}$ <u>den naturlige def. mengden.</u>

<u>Eks.</u> $F(x_1, x_2) = \dfrac{x_1^2 + x_2^2}{(x_1 - 1)^2}$

har <u>naturlig def. mengde</u> $\{(x_1, x_2) : x_1 \neq 1\}$,

og <u>bilde $\mathbb{R}_{\geq 0} = [0, \infty)$.</u>
