11.4 Krumning, torsjon og Frenetrammen

Krumning forholder seg til fart, (omtrent) som den andrederiverte til den forstederiverte.

Husk: $\gamma : I \to \mathbb{R}^n$, glatt, $\exists |\dot\gamma| \neq 0$

$$T(t) = \frac{v(t)}{|v(t)|} = \frac{\dot\gamma(t)}{|\dot\gamma(t)|}$$ enhetstangent

Def. Krumningen $\kappa \overset{\text{def.}}{=} \left|\frac{dT}{ds}\right|$

buelengdep.

Hva er $\frac{dT}{ds}$?

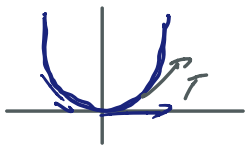
Den deriverte av tangenten når hastigheten er konst 1 langs kurven.

Merk: $\frac{ds}{dt} = |\dot\gamma(t)| = |v(t)| \Rightarrow \boxed{\frac{dt}{ds} = \frac{1}{|v(t)|}}$

$$\boxed{\text{Så } \kappa = \left|\frac{dT}{ds}\right| \underset{\text{kj.r.}}{=} \left|\frac{dT}{dt}\boxed{\frac{dt}{ds}}\right| = \frac{1}{|v(t)|}\left|\frac{dT}{dt}\right|}$$

Eks. ($\subset \mathbb{R}^2$):

(i) parabel $\gamma(t) = (t, t^2)$, $t \in \mathbb{R}$



$\dot\gamma(t) = v(t) = (1, 2t)$, $|\dot\gamma(t)| = |v(t)| = \sqrt{1+4t^2} \neq 0$

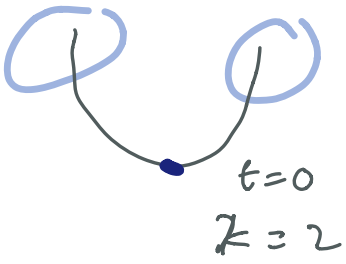
$T = \frac{v}{|v|} = \frac{(1,2t)}{(1+4t^2)^{\frac12}}$

$$\left|\frac{dT}{dt}\right| \overset{\text{MA1102}}{=} \left|\frac{(-4t, 2)}{(1+4t^2)^{\frac32}}\right| = \frac{(4(4t^2)+4)^{\frac12}}{(1+4t^2)^{\frac32}}$$

$$= \frac{2}{1+4t^2}.$$

Så $\kappa = \left|\frac{dT}{dt}\right| \frac{1}{|v|} = \frac{2}{1+4t^2} \cdot \frac{1}{(1+4t^2)^{\frac12}} = \frac{2}{(1+4t^2)^{\frac32}}$

$\kappa \to 0$ når $t \to \pm\infty$.

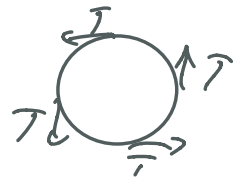


(ii) sirkel $\gamma(\theta) = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$.

$|\dot\gamma(\theta)| = |(-\sin\theta, \cos\theta)| = 1 \quad \forall\theta$

Så $\theta = s$ er buelengden.

$\Rightarrow \quad T = \frac{v}{|v|} = v = (-\sin\theta, \cos\theta)$



og $\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{d\theta}\right| = \left|\frac{dv}{d\theta}\right| = |(-\cos\theta, -\sin\theta)| = 1.$

Enhetssirkelen har konstant krumning 1.

Enhetsnormalen

Ettersom $|T(t)| = 1 \quad \forall t$, følger

$T \cdot T' = 0$, dvs T' normal mot T.

Def. Enhetsnormalen

$$N \overset{\text{def.}}{=} \frac{dT/ds}{|dT/ds|} = \frac{1}{\kappa}\frac{dT}{ds}.$$

$\frac{ds}{dt} = |v| > 0$

Ved kjerneregelen:

$$N = \frac{\frac{dT}{dt}\cdot\frac{dt}{ds}}{\left|\frac{dT}{dt}\cdot\frac{dt}{ds}\right|} = \frac{dT/dt}{|dT/dt|}.$$

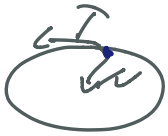
κ er til N, som $|v|$ er til T.

Eks. (i) parabelen $T = \frac{(1, 2t)}{\sqrt{1+4t^2}}$ og

$N = \frac{(-2t, 1)}{\sqrt{1+4t^2}}$ $(-4t, 2)$ med lengde 1.

$T \cdot N = \frac{-2t + 2t}{1+4t^2} = 0.$

(ii) sirkelen $\boxed{\begin{aligned} T &= (-\sin\theta, \cos\theta) \\ N &= (-\cos\theta, -\sin\theta) \end{aligned}}$



$T \perp N$

Så i $\mathbb{R}^2$ er $\{T, N\}$ et lokalt (lengs kurven) koordinatsystem (ortogonal bas).

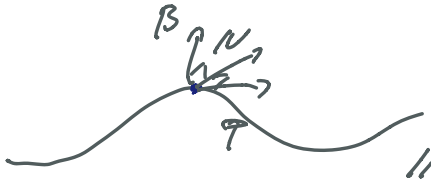
I $\mathbb{R}^3$ kan vi lage oss binormalen

$\boxed{B \overset{\text{def.}}{=} T \times N}$ $B \perp T$, $B \perp N$

$|B| = |T \times N| = 1.$

torsjon

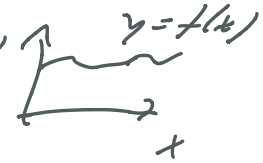
$\frac{dB}{ds} = -\tau N$



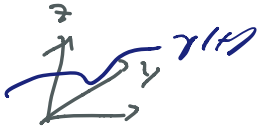
Frenet-rammen $\{T, N, B\}$ 3-dim lokalt koord.syst. langs kurven.

12.1 Funksjoner av flere variabler

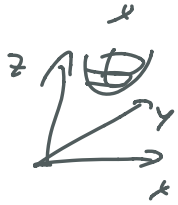
Reelle funksjoner: $f: \mathbb{R} \to \mathbb{R}$



Kurver: $\gamma: \mathbb{R} \to \mathbb{R}^n$



Nå: $F: \mathbb{R}^n \to \mathbb{R}$



$z = F(x,y)$

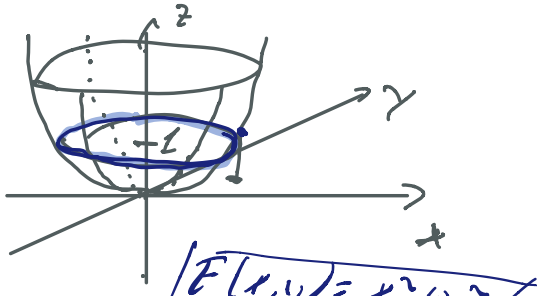
Enkleste eksempler:

$F: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto x^2 + y^2$

Graf: $\mathbb{R}^3$:

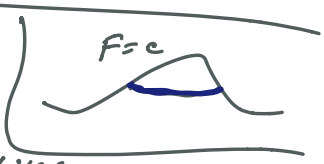
$\{(x,y,z): z = x^2 + y^2\}$



$F(x,y) = x^2 + y^2$

Paraboloide (se 10.5)

Grafen er en **overflate**.



Kurvene $\{F(x,y) = c\}$ er **nivåkurver**.

Generelt $\{F(x_1, x_2, \ldots, x_n) = c\}$ **nivåmengde**.

En funksjon $F: U \subset \mathbb{R}^n \longrightarrow F(U) \subset \mathbb{R}$

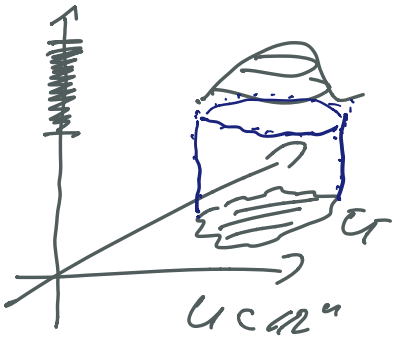
$$x = (x_1, x_2, \dots, x_n) \longmapsto F(x) = F(x_1, \dots, x_n)$$

Har:

(i) en definisjonsmengde U. (også domene)

(ii) kodomene $\mathbb{R}$

(iii) Et bilde $\{F(x) : x \in U\} \subset \mathbb{R}$.



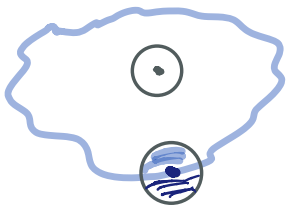
Dersom U ikke er gitt, er $\{x \in \mathbb{R}^n : F(x) \in \mathbb{R}\}$ den naturlige definisjonsmengden.

Eks. $F(x_1, x_2) = \dfrac{x_1^2 + x_2^2}{(x_1 - 1)^2}$

har naturlig definisjonsmengde $\{(x_1, x_2) : x_1 \neq 1\}$,

og bilde $\mathbb{R}_{\geq 0} = [0, \infty)$.

Åpne og lukkede mengder



Def. Et punkt $x_0 \in \mathbb{R}^n$ er:

- et indre punkt i U dersom

$\exists \varepsilon > 0$; $B_\varepsilon(x_0) \subset U$.

$B_\varepsilon(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < \varepsilon \}$.

- et randpunkt til U dersom

$\forall \varepsilon > 0$: $B_\varepsilon(x_0) \cap U \neq \emptyset$

men $B_\varepsilon(x_0) \not\subset U$.

- Mengden av randpunkter til U kalles randen til U, ∂U.

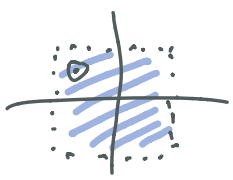
- $\overline{U} = U \cup \partial U$ kalles tilluknningen av U

- Mengden U er åpen dersom alle punkter i U er indre punkter: $\partial U \cap U = \emptyset$.

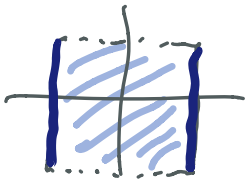
- Lukket dersom alle randpunkter tilhører U: $\overline{U} = U$.

Eks. $\{ |x_1| < 1, |x_2| < 1 \}$

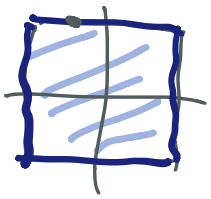
åpen, ikke lukket



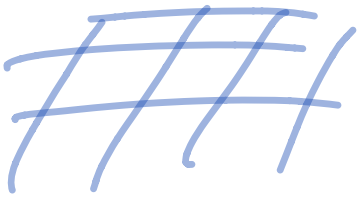
- $\{|x_1| \leq 1,\ |x_2| < 1\}$
  ikke åpen, ikke lukket



- $\{|x_1| \leq 1,\ |x_2| \leq 1\}$
  lukket, ikke åpen



- $\mathbb{R}^2$ er åpen og lukket
  (tinner ingen randpunkter)



Grenseverdier og kontinuitet

Def. (i) $F(x)$ har grenseverdi L i $x_0 \in \overline{U}$, $\lim_{x \to x_0} F(x) = L$, dersom

$\forall \varepsilon > 0 \quad \exists \delta > 0$;

$|\underbrace{F(x) - L}_{\mathbb{R}}| < \varepsilon$ når $|\underbrace{x - x_0}_{\mathbb{R}^n}| < \delta$ og $x \in U$.

obs! x_0 trenger ikke tilhøre U

(ii) F er kont. i $x_0 \in U$ dersom

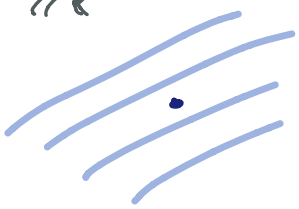
$\exists \lim_{x \to x_0} F(x) = F(x_0)$

x_0 tilhører U!

$\uparrow L$

Eks. $F: U \subset \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$

$(x_1, x_2) \mapsto \dfrac{x_1^4 - x_2^4}{x_1^2 + x_2^2}$



Grenseverdi i $(0,0)$?

$$F(x_1, x_2) = \frac{(x_1^2 + x_2^2)(x_1^2 - x_2^2)}{x_1^2 + x_2^2} = x_1^2 - x_2^2 \to 0 \; ?$$

$\forall (x_1, x_2) \in U$.

Så $|F(x_1,x_2) - 0| = |x_1^2 - x_2^2| \overset{\Delta\text{-ulik}}{\leq} x_1^2 + x_2^2 \overset{!}{<} \varepsilon$ $\quad \delta^2$

når $|(x_1, x_2) - (0,0)| = \underbrace{(x_1^2 + x_2^2)^{\frac{1}{2}}}_{\delta} < \delta$ $\quad |a-b| \leq |a| + |b|$

dersom $\delta < \sqrt{\varepsilon}$

$$\Rightarrow \boxed{\lim_{(x_1,x_2)\to(0,0)} F(x_1, x_2) = 0}$$

$\square$

Teorem Summer og produkter av kont. funksj. er kontinuerlige.

Bevis $F, G: U \subset \mathbb{R}^n \to \mathbb{R}$ kont.

La $x_0 \in U$, $\varepsilon > 0$.

(i) $|(F+G)(x) - (F+G)(x_0)|$

$= |F(x) - F(x_0) + G(x) - G(x_0)|$

$\leq$ (Δ-olik.) $|F(x)-F(x_0)| + |G(x)-G(x_0)| = (*)$

F, G kont $\Rightarrow \exists \delta$: $(*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

for $|x - x_0| < \delta$ og $x \in U \Rightarrow F+G$ kont i x_0

x_0 vilkårlig : $U \Rightarrow$ $F+G$ kont på U.

(ii) $|(F\cdot G)(x) - F\cdot G(x_0)| = |F(x)G(x) - F(x_0)G(x_0)|$

$= |(F(x) - F(x_0))G(x) + F(x_0)(G(x) - G(x_0))|$

$(G(x) - G(x_0)) + G(x_0)$

$\leq$ (Δ-olik) $|F(x)-F(x_0)|\,|G(x)-G(x_0)|$

$+ |F(x)-F(x_0)|\,|G(x_0)|$

$+ |G(x)-G(x_0)|\,|F(x_0)| = (**)$

$\varepsilon < 1$

Vels δ : $|F(x)-F(x_0)| < \frac{\varepsilon}{3 \max\{|F(x_0)|, |G(x_0)|, 2\}}$

$|G(x)-G(x_0)| <$ — " —

da $|x - x_0| < \delta$.

$\Rightarrow (**) < \frac{\varepsilon}{3}\cdot\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \overset{\varepsilon<1}{\leq} 3\cdot\frac{\varepsilon}{3} = \varepsilon.$

$\Rightarrow$ F.G kont i x_0 $\overset{x_0 \text{ vilk.}}{\Rightarrow}$ F.G kont på U.

Eks. Vet $x_i \mapsto x_i^2$ kont $\mathbb{R} \to \mathbb{R}$

$\Rightarrow (x_1, x_n) \mapsto x_i^2$ kont $\mathbb{R}^n \to \mathbb{R}$

samme dat

$\Rightarrow$ $x_i^2 - x_n^2$ kont $\mathbb{R}^n \to \mathbb{R}$

Teorem

$\Rightarrow$ $\frac{x_i^4 - x_n^4}{x_i^2 + x_n^2} = x_i^2 - x_n^2$ kont på $\mathbb{R}^n$.

Eks. $(x_1, x_n) \neq (0,0)$

12.3 Partielt deriverte

For $f: \mathbb{R} \to \mathbb{R}$ er

(i) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

(ii) $f(x+h) = f(x) + hf'(x) + h\underbrace{\varepsilon(h)}_{\to 0 \text{ da } h\to 0}$

litet

ekvivalente def. for den deriverte.

I høyere dim. gir disse to forskjellige begrepene av den deriverte.

Def. La $F: U \subset \mathbb{R}^n \to \mathbb{R}$, $x_0 \in U$, $v \in \mathbb{R}^n$, $|v| = 1$.

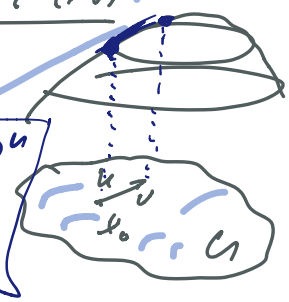
(i) Den retningsderiverte av F i punktet x_0 i retning v er:

$$\partial_v F(x_0) \overset{\text{def.}}{=} \lim_{h \to 0} \frac{F(x_0 + hv) - F(x_0)}{h}$$

(dersom grensen eksisterer)

$h \in \mathbb{R}$, $v \in \mathbb{R}^n$

$\partial_v F(x_0) \in \mathbb{R}$



'hvor raskt vokser F i retning v fra x_0?'

(Kan også skrive $\frac{\partial f}{\partial v}$, $D_v F$, F'_v, ...)

- De enkleste retningsderivatene er når $v = e_j$, (basisvektor i $\mathbb{R}^n$). Disse kalles partielt deriverte.

Eks. $F(x_1, x_n) = x_1^2 \sin(x_n)$

$$v = (1, 0) \Longrightarrow \partial_v F(x_1, x_2) = \partial_{x_1} F(x_1, x_2)$$
$$= 2x_1 \sin(x_2).$$

$$v = (0, 1) \Longrightarrow \partial_v F(x_1, x_2) = \partial_{x_2} F(x_1, x_2)$$
$$= x_1^2 \cos(x_2).$$

$\partial_{x_j} F$ skriver også

$\frac{\partial F}{\partial x_j}$, F'_{x_j}, $D_{x_j} F$, ...

'vanlig' derivatet