$\tan y = \frac{v}{u} \iff y = \arctan\left(\frac{v}{u}\right) + k\pi$

+ må ta hensyn til $u = 0$.

14.4 Variabelsubstitusjon i dobbeltintegraler

Teorem Anta at $\iint_{\bar{U}} dA$ eksisterer. Jakobiden.

La $\Phi : \bar{U} \to \mathbb{R}^2$ være C^1 med $\det(D\Phi) \neq 0$ på $\bar{U}$.

Da gjelder:

$$\boxed{\iint_{\Phi(\bar{U})} f \, dA = \iint_{\bar{U}} f \circ \Phi \, \underbrace{|D\Phi|}_{|\det(D\Phi)|} \, dA}$$

Eks. (i $\mathbb{R}$!) $\bar{U} = [0,1]$, $\Phi(x) = 5x + 2$

$D\Phi = 5 \neq 0 \Rightarrow \exists \Phi^{-1} \in C^1$ (lokalt)

$\Phi(\bar{U}) = [2,7]$

$$\boxed{\int_{[2,7]} f(y)\,dy = \left[\begin{array}{l} y = 5x+2 \\ dy = 5\,dx \end{array}\right] = \int_{[0,1]} f(5x+2)\,5\,dx}$$

obs! I 1D dukker en orientering opp, som den i $dy = -dx$ $(y = -x)$.

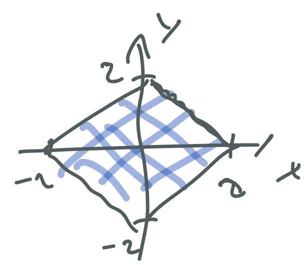
Denne er ikke synlig i $\boxed{}$, fordi vi bruker absoluttbeløp og mengder.

Sml. $x = -y$ $\quad \left|\frac{dx}{dy}\right| = 1.$ $\quad \boxed{\int_{[-1,0]} f(y)\,dy = \int_{[0,1]} f(x)\,1\,dx \quad \uparrow}$

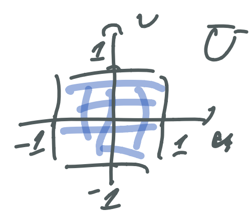
Med orientering: $\displaystyle\int_{-1}^{0} f(y)\,dy = -\int_{1}^{0} f(x)\,dx = \int_{0}^{1} f(x)\,dx.$

Eks i $\mathbb{R}^2$: skalert rotasjon

Gitt $\underbrace{V}_{\Phi(U)} = \{-2 \le x+y \le 2,\ -2 \le x-y \le 2\}$



Forenkle? La $\begin{cases} u = \frac{x+y}{2} \\ v = \frac{x-y}{2} \end{cases}$



$\begin{cases} x = u+v \\ y = u-v \end{cases}$ $\quad \Phi : (u,v) \longmapsto (x,y)$

$$|D\Phi| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2.$$

Så: $\displaystyle\iint_V dx\,dy = 2\iint_U du\,dv.$

Eks. 2 Polarkoord.

$$\Phi : (r,\theta) \mapsto (x,y)$$

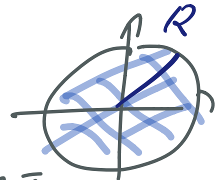
$$\boxed{dx\,dy = r\,dr\,d\theta}$$

$x = r\cos\theta$, $y = r\sin\theta$

$$|D\Phi| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \neq 0$$

utenfor origo.

$\bar{V} = \{x^2+y^2 \leq R^2\}$

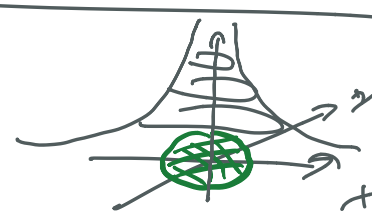


$\bar{U} = \{0 \leq \theta < 2\pi,\ 0 < r \leq R\}$

$$\iint_{x^2+y^2\leq R^2} dx\,dy = \iint_{\bar{U}} r\,dr\,d\theta = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = 2\pi \frac{r^2}{2}\Big|_0^R$$

$$= \underline{\underline{\pi R^2}}.$$

Eks 3 Uegentlige integraler



$$\iint_{1\leq x^2+y^2\leq R^2} \frac{dx\,dy}{x^2+y^2} = \int_0^{2\pi}\int_1^R \frac{r\,dr\,d\theta}{r^2}$$

$$= 2\pi \ln r\Big|_1^R = 2\pi\ln(R) \longrightarrow \infty \quad \text{da } R\to\infty.$$

Integralet divergent.

$$\iint_{1\leq x^2+y^2\leq R^2} \frac{dx\,dy}{(x^2+y^2)^{3/2}} = \int_0^{2\pi}\int_1^R \frac{r\,dr\,d\theta}{r^3} = 2\pi\left[-\frac{1}{r}\right]_1^R$$

$$= 2\pi\left(1-\frac{1}{R}\right) \longrightarrow \underline{\underline{2\pi}}.$$

da $R \to \infty$. Integralet konvergent.

Nå: trippelintegraler 14.5, 14.6 + 10.6

$$\iiint_V dV = \iiint_V dx\,dy\,dz \quad \text{måler \underline{volumet} til } V.$$

- Samme def. (Riemannsummer) som for $\mathbb{R}$, $\mathbb{R}^2$, men i små kuber / legemer i stedet.

- $\iiint_V f(x,y,z)\,dx\,dy\,dz$ måler ofte en størrelse over et legeme, f.eks. densitet, intensitet, varme:
  '$\iiint_V$ densitet = vekt.'

- Som tidligere, kan trippelintegraler beregnes gjennom iterert integrasjon:

'Fubini': f, g, h kontinuerlige,
$V = \{ g(x,y) \le z \le h(x,y),\ (x,y) \in \tilde{V} \}$

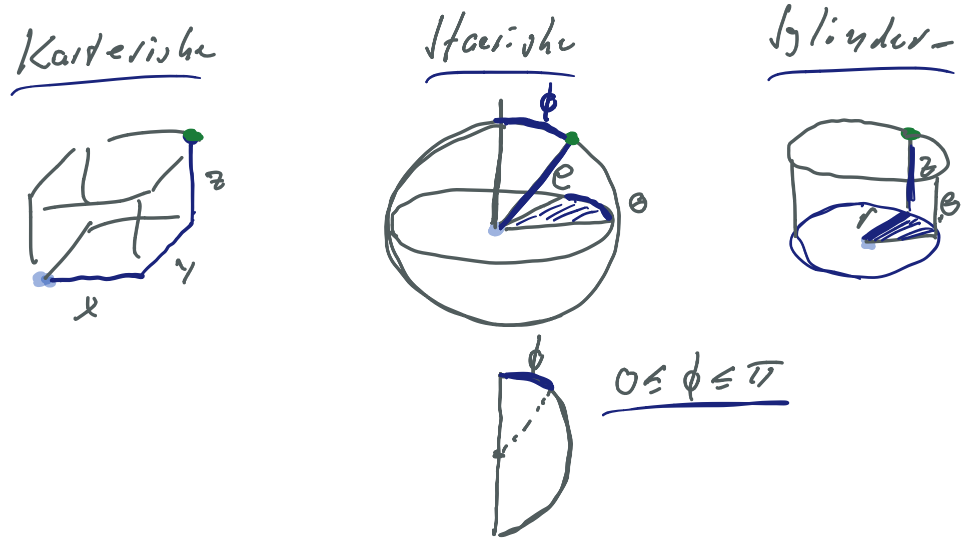
$$\Rightarrow \quad \iiint_V f\,dV = \iint_{\tilde{V}} \left(\int_{g(x,y)}^{h(x,y)} f(x,y,\cdot)\,dz \right) dA$$

- Variabelbytte som i to dimensjoner:

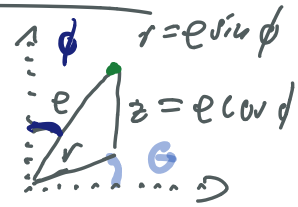
$$\boxed{\iiint\limits_{\Phi(\bar V)} f \underbrace{dx\,dy\,dz}_{dV} = \iiint\limits_{\bar V} f\circ\Phi\, \underbrace{|D\Phi|}_{3\times 3-\text{matrise}} \underbrace{du\,dv\,dw}_{dV}}$$

Vanlige variabelrepresentasjoner i $\mathbb{R}^3$:

sylindriske og sfæriske koord.
(kulekoord.)



$$\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = z = \rho\cos\phi \end{cases}$$



$r = \sqrt{x^2+y^2}$

$\rho = \sqrt{x^2+y^2+z^2}$

$\partial_r \qquad \partial_\theta \qquad \partial_z$

$$|D\Phi|_{\text{syl.}} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

$$\Rightarrow \boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz} \quad (\text{polark.} + z)$$

$$|D\Phi|_{\text{sfær.}} = \begin{array}{c} \begin{array}{ccc} \quad\partial\rho\quad & \qquad\partial\phi\qquad & \qquad\partial\theta\quad \end{array} \\ \begin{vmatrix} \cos\theta\sin\phi & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\theta\sin\phi & \rho\sin\theta\cos\phi & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \end{array}$$

$$= 0 + \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin\phi\sin^2\phi\sin^2\theta$$

$$- \left[-\rho^2\sin^3\phi\cos^2\theta + 0 - \rho^2\sin\phi\cos^2\phi\sin^2\theta\right]$$

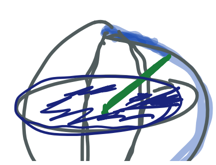
$$= \rho^2\left[\sin^2\phi\sin\phi + \cos^2\phi\sin\phi\right] = \rho^2\sin\phi$$

$$\Rightarrow \boxed{dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

obs $0 < \phi < \pi$
$\sin\phi > 0$.

Eks. volum av en kule med radius $R > 0$.

$$\iiint_{x^2+y^2+z^2 \le R^2} dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\overset{\text{stor.}}{\pi}} \sin\phi \, d\phi\right)\left(\int_0^{R} \rho^2 \, d\rho\right)$$

$0 \le \phi \le \pi$

$0 \le \theta < 2\pi$

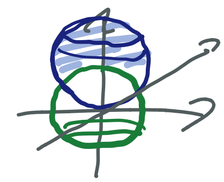
$0 \le \rho \le R$

$$= 2\pi \underbrace{\left[-\cos\phi\right]_0^{\pi}}_{2} \frac{\rho^3}{3}\Big|_0^R = \frac{4\pi}{3}R^3.$$

◻

Eks: integral over $B_1(0,0,1) \setminus B_1(0,0,0)$

$$V = \{(x,y,z) : x^2+y^2+(z-1)^2 < 1 \text{ og } x^2+y^2+z^2 \ge 1\}$$



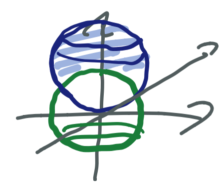
$0 \le \theta < 2\pi$

$S_{z=1}: \; x^2+y^2+(z-1)^2 = 1$

$$\iff \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) + (\rho\cos\phi \cancel{-1})^2 = \cancel{1}.$$

$$\iff \rho^2 \underbrace{(\sin^2\phi+\cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\iff \begin{cases} \cancel{\rho = 0 \text{ eller}} \\ \boxed{\rho = 2\cos\phi} \;(\text{inkl. } \rho = 0) \end{cases}$$



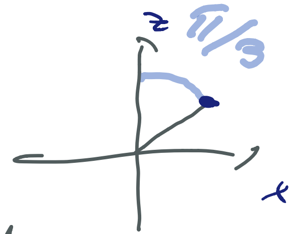
$S_{z=0}: \; x^2+y^2+z^2 = 1 \iff \boxed{\rho = 1.}$

Så: $1 \le \rho < 2\cos\phi$

$0 \le \phi < \frac{\pi}{3}$

Fine skjæringen!

Skjæring: $\cos\phi = \frac{1}{2} \iff \phi = \frac{\pi}{3}$.



Integral: $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$= 2\pi \int_0^{\pi/3} \sin\phi \left[\frac{\rho^3}{3}\right]_1^{2\cos\phi} d\phi$$

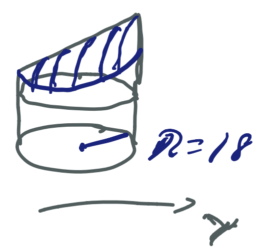
$$= \frac{2\pi}{3} \int_0^{\pi/3} (8\sin\phi\cos^3\phi - \sin\phi)\, d\phi$$

$$= \frac{2\pi}{3} \Big[-2\cos^4\phi + \cos\phi\Big]_0^{\pi/3}$$

$$= \frac{2\pi}{3} \left[-\frac{1}{8} + \frac{1}{2} + 2 - 1\right] = \frac{11}{12}\pi.$$

Eks. opps. 5 14/5 2002

Beregn volumet til en kirke med vegger $x^2+y^2=(18)^2$, tak $z = 20 - \frac{x^2}{25} + \frac{y}{2}$ for $x^2+y^2 \le 400$ og gulv ved $z=0$.



Løsn. $D = \{(x,y,z) : 0 \le \overset{r^2}{x^2+y^2} \le (18)^2,$
$0 \le z \le 20 - \frac{x^2}{25} + \frac{y}{2}\}$ $-\pi \le \theta \le \pi$

$$\iiint_D dV = \iint_{x^2+y^2\le 18^2} \left(\int_0^{20-\frac{x^2}{25}+\frac{y}{2}} dz\right) dx\,dy = \left[\begin{array}{l}\text{sylinder}\\ \text{koord.}\end{array}\right]$$

$$= \int_{-\pi}^{\pi}\int_0^{18} \Big(20 - \underbrace{\frac{r^2\cos^2\theta}{25}}_{2\pi r^2/2\cdot 25} + \underbrace{\frac{r\sin\theta}{2}}_{=0}\Big)\, r\,dr\,d\theta = (*)$$

sin odde $\Rightarrow \int_{-\pi}^{\pi}\sin\theta\,d\theta = 0$

cos² like med $\int_{-\pi}^{\pi}\cos^2\theta\,d\theta = \int_{-\pi}^{\pi}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\cdot 2\pi$

$$(*) = \cancel{2}\pi\left[20\frac{r^2}{\cancel{2}} - \frac{r^4}{4\cdot\cancel{2}\cdot 25}\right]_0^{18}$$

$$= \pi\left[20\cdot 18^2 - \frac{18^4}{100}\right] = 5430{,}24\pi.$$

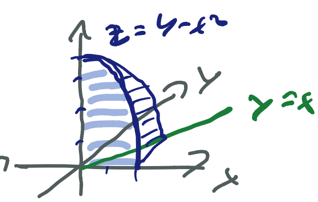
Volumet.

Eks. opps. 4 8.8 2001

Beregn $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin(2z)}{z-4}\,dy\,dz\,dx = (**)$

Løsn. $0\le y\le x$

$$\begin{cases}0\le z\le 4-x^2\\ 0\le x\le 2\end{cases} \iff \begin{array}{l}0\le z\le 4\\ 0\le x\le\sqrt{4-z}\end{array}$$



Kaster om int. rekkefølgen!

1. $$\int_0^x \frac{\sin(2z)}{z-4}\,dy = \frac{x\sin(2z)}{z-4}$$

2. **Reparametrisér:**

$$\frac{1}{z-4}\cdot\frac{4-z}{2}$$

$$(*) = \int_0^4\int_0^{\sqrt{4-z}} \frac{x\sin(2z)}{z-4}\,dx\,dz$$

$$= \int_0^4 \frac{\sin(2z)}{z-4}\underbrace{\left(\int_0^{\sqrt{4-z}} x\,dx\right)}_{\frac{x^2}{2}\Big|_0^{\sqrt{4-z}} = \frac{4-z}{2}}dz = -\frac{1}{2}\int_0^4 \sin(2z)\,dz$$

obs! tegn!

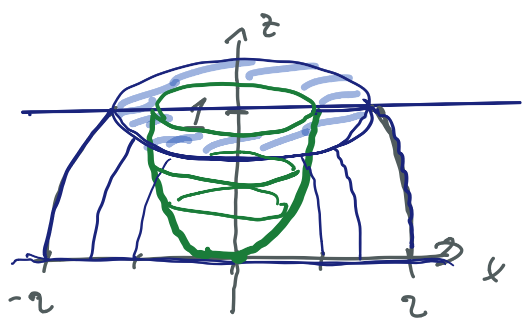
$$= \frac{1}{4}(\cos(8)-1).$$

Eks. opgs. 3 19.12 2006

R avgrenset av $x^2+y^2+z^2=4$, $z=x^2+y^2$,
kuleskall, paraboloide

$z=0$ og $z=1$.
plan, plan

a) Skissér R.



Beräkna $V(R)$.

$$R = \{0 \le z \le 1,\ z \le x^2+y^2 \le 4-z^2\}$$

$$\iiint_R dV = \begin{bmatrix} \text{cyl.} \\ \text{koord.} \end{bmatrix} = \int_0^{2\pi}\int_0^1\int_{\sqrt{z}}^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta$$

$$= \cancel{2}\pi \int_0^1 \left[\frac{r^2}{\cancel{2}}\right]_{\sqrt{z}}^{\sqrt{4-z^2}} dz = \pi \int_0^1 \left((4-z^2)-z\right) dz = \frac{19\pi}{6}.$$

$4-z^2 \ge 0$

$0 \le z \le 1$