MA1103 Exercise Set 7

Norwegian University of Science and Technology

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IMPORTANT: The deadline for this exercise is Friday March 5. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

Problem $\boxed{1}$

Old exam problem. Given a parametrized curve $\gamma : [0, 2\pi] \to \mathbb{R}^2,\ t \mapsto \gamma(t) = (\cos t, \sin t)$, and the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto (x^2 + y^2)^{\frac{3}{2}} + (x^2 + y^2)^{\frac{5}{2}},$$

calculate

$$\int_\gamma \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s$$

where $\mathbf{T}$ denotes the unit tangent vector of γ and $\mathbf{F} = \nabla f$.

Problem $\boxed{2}$

Old exam problem. Suppose that a certain projectile which we would like to study, moves along a curve C in space $\mathbb{R}^3$ such that the position of the projectile at time $t \geq 0$ is given by the vector-valued function $\mathbf{c}(t) = \left(t, \frac{2}{3}\sqrt{2}t^{3/2}, \frac{1}{2}t^2\right)$.

a) If the projectile starts moving at $t = 0$ seconds and continues to move, and the speed is given in meters per second, how many meters have it traversed at time 10 seconds?

b) Assume the temperature in space at the point (x, y, z) is given by $T(x, y, z) = x^2 + xz + y$. Determine then the rate of change in temperature the projectile will experience at time $t = 1$.

Problem $\boxed{3}$

Find the maximum and minimum values of the function

$$(x, y) \mapsto f(x, y) = xy - y^2$$

on the closed unit disk $\overline{\mathbb{D}} = \{\mathbf{v} \in \mathbb{R}^2 : |\mathbf{v}| \leq 1\}$.

Problem $\boxed{4}$

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a given vector-valued function and let $K \subset \mathbb{R}^n$ be compact. The image of K under f is given by

$$f(K) = \{f(x) \in \mathbb{R}^m : x \in K\}.$$

Show that if f is continuous, then $f(K)$ is compact, that is, closed and bounded. Conclude the extreme-value theorem.

Hint: Previous exercises.

Problem $\boxed{5}$

Consider the system of equations

$$u = x \cos y \quad \text{and} \quad v = x \sin y.$$

Show that near the point (x_0, y_0) with $x_0 \neq 0$, (x, y) can be expressed as a differentiable function of (u, v) and compute $\frac{\partial x}{\partial u}(u_0, v_0)$ and $\frac{\partial x}{\partial v}(u_0, v_0)$.

Hint: Verify the two conditions of The Inverse Function Theorem first, then use it to compute the partial derivatives.

Problem $\boxed{6}$

Let $f : \mathbb{R}^2 \supseteq U \to \mathbb{R}, (x, y) \mapsto f(x, y)$ be a given twice continuously differentiable real-valued function of 2 variables defined on some open subset $U \subseteq \mathbb{R}^2$. Its Laplacian Δf, in cartesian coordinates (x, y), is given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Using the chain rule, determine the Laplacian of f in polar coordinates (r, θ). How does Δf vary if the coordinate plane is rotated by an angle α counter-clockwise?

Problem $\boxed{7}$

Given that $F(x, y, z) = 0$, let $\mathbf{p}_0 = (x_0, y_0, z_0)$ and $F(\mathbf{p}_0) = 0$. Assume $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ are continuous and not equal to 0 at $\mathbf{p}_0$.

a) Show that near $\mathbf{p}_0$, we can write

$$x = f_1(y, z), \quad y = f_2(x, z), \quad z = f_3(x, y),$$

where f_1, f_2, f_3 are differentiable functions on their variables.

b) Prove that

$$\frac{\partial f_1}{\partial y}(\mathbf{p}_0)\frac{\partial f_2}{\partial z}(\mathbf{p}_0)\frac{\partial f_3}{\partial x}(\mathbf{p}_0) = -1.$$

Hint: Check the three conditions of The Implicit Function Theorem first, then use it to calculate the partial derivatives one by one from the equation $F(x, y, z) = 0$.

Problem $\boxed{8}$

Let $\overline{\mathbb{D}} := \{\mathbf{v} \in \mathbb{R}^2 : |\mathbf{v}| \leq 1\}$ denote the closed unit disc in the plane and let

$$f : \overline{\mathbb{D}} \to \mathbb{R}, \quad (x, y) \mapsto f(x, y) = (x + y)e^{-x^2-y^2}$$

be a given real-valued function defined on $\overline{\mathbb{D}}$.

(a) Argue that f assumes its global maximum and minimum values on $\overline{\mathbb{D}}$.

(b) Determine the global maximum and minimum values of f and give the corresponding points in $\overline{\mathbb{D}}$ at which f attains these values.

Hint: It is possible to solve this *without* using Lagrange multipliers (it is also possible to solve this *using* Lagrange multipliers).