MA1103 Exercise Set 5

Norwegian University of Science and Technology

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IMPORTANT: The deadline for this exercise is Friday Feb 19. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

Problem $\boxed{1}$

Find equations of the tangent plane and normal line to the function

$$f : (x, y) \mapsto \frac{x-y}{x+y}$$

defined on its natural domain of definition in $\mathbb{R}^2$, at the point $(1, 1)$.

Problem $\boxed{2}$

Find the gradient of the function $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto z + x^2 y + e^{y\cos(xz)}$.

Problem $\boxed{3}$

For the function

$$f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 - y^2,$$

find:

(a) the gradient of the given function at the point $(2, -1)$,

(b) an equation of the plane tangent to the given function at the point $(2, -1)$, and

(c) an equation of the straight line tangent, at the point $(2, -1)$, to the level curve of the given function passing through this point.

Problem $\boxed{4}$

Consider the surface $z = f(x, y)$ in $\mathbb{R}^3$ where f is the function

$$f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto e^{x^2+y}.$$

For precisely one $c \in \mathbb{R}$, the point $p = (2, -4, c)$ lies on the surface. Determine c and the tangent plane at p.

Problem $\boxed{5}$

Find the rate of change of the function

$$f : (x, y) \mapsto \frac{x}{1+y}$$

defined on its natural domain of definition in $\mathbb{R}^2$, at the origin in the direction $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$ where we use the notation $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$.

Problem $\boxed{6}$

Let $B_r(\mathbf{0})$ be an open ball centred at the origin with radius $r > 0$ in $\mathbb{R}^n$ and suppose that $f : B_r(\mathbf{0}) \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}), \mathbf{x} = (x_1, \ldots, x_n)$, is a given real-valued differentiable function on $B_r(\mathbf{0})$. For $t \in [0, 1]$, consider the function

$$h : t \mapsto f(t\mathbf{x}).$$

Differentiate h with respect to t using the chain rule, and integrate the result over the interval $[0, 1]$ using the fundamental theorem of calculus, to conclude that we can write

$$f(\mathbf{x}) = f(\mathbf{0}) + \sum_{j=1}^{n} g_j(\mathbf{x}) x_j$$

for some continuous real-valued functions g_j defined on $B_\varepsilon(\mathbf{0})$ such that $g_j(\mathbf{0}) = \frac{\partial f}{\partial x_j}(\mathbf{0})$.

Remark: this is a kind of Taylor formula in several variables.

Problem $\boxed{7}$

In what directions at the point $(2, 0)$ does the function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto xy$ have rate of change -1? Are there directions in which the rate is -3?

Problem $\boxed{8}$

Determine the directional derivative at the point p in the direction $\mathbf{v}$ of the given functions.

a) $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto z \sin(xy), p = \left(\frac{\pi}{2}, 1, 0\right), \mathbf{v} = \frac{1}{\sqrt{5}}(2, 0, -1)$.

b) $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto 100 - 2x^2 - 3y^2, p = (2, 3)$ and $\mathbf{v}$ is the direction in for which the directional derivative is greatest.

Problem $\boxed{9}$

Prove that if a real-valued function f is differentiable at a point $\mathbf{a} \in \mathbb{R}^n$, then f is continuous at $\mathbf{a}$.

Problem $\boxed{10}$

Using the definition of the derivative as a linear map, prove the proposition below. You may take for granted the following fact: *the derivative if it exists, is unique.*

Proposition: Let $U \subset \mathbb{R}^n$ be open, and let $f, g : U \to \mathbb{R}$ two real-valued functions differentiable at $a \in U$. Let $f'(a)$ and $g'(a)$ denote the derivatives at a of f and g respectively. Then fg is differentiable at a with derivative

$$(fg)'(a) = f(a)g'(a) + g(a)f'(a).$$

Hint: consider first the case $n = 1$. Mimic this and apply to the uniqueness of the derivative. The triangle inequality and Problem $\boxed{9}$ may also be useful.