MA1103 Exercise Set 4

Norwegian University of Science and Technology

January 29, 2021

IMPORTANT: The deadline for this exercise is Friday Feb 12. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

Problem $\boxed{1}$

The triangle inequality says that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$

a) Prove that the triangle inequality gives the *reverse triangle inequality*:

$$||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} - \mathbf{y}|, \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

b) Let $f : \mathbb{R}^n \to [0, \infty)$ be defined by $f(\mathbf{x}) = |\mathbf{x}|$. Argue that f is continuous on all of $\mathbb{R}^n$.

Problem $\boxed{2}$

Transform the given polar equation to rectangular coordinate, and identify which type of the curve it is.

$$r = \sin\theta + \cos\theta.$$

Problem $\boxed{3}$

Determine the closest point in the plane $2x - 3y + z = 0$ to the point $(1, -2, 1)$.

Problem $\boxed{4}$

Write the Product Rule for $\frac{d}{dt}\Big(\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\Big)$, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are 3-dimensional vector-valued functions of a single real variable t, and have continuous derivatives of all required orders.

Problem $\boxed{5}$

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ be pairwise orthogonal vectors. That is, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. Show that these are then linearly independent.

Problem $\boxed{6}$

Evaluate the indicated limit or explain why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2}.$$

Problem $\boxed{7}$

Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbf{f}, \mathbf{g} : I \to \mathbb{R}^3$ be differentiable vector-valued functions such that $\mathbf{f}(t) \cdot \mathbf{g}(t) = 0$ for all $t \in I$. Let $\mathbf{h} = \mathbf{f} \times \mathbf{g}$ and suppose furthermore that $|\mathbf{f}(t)| = |\mathbf{g}(t)| = 1$ for all $t \in I$. Argue that there are scalar-valued functions $a, b, c : I \to \mathbb{R}$ such that

$$\frac{d}{dt}\begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \\ \mathbf{h} \end{bmatrix},$$

on I. **Hint:** Problem $\boxed{5}$ together with the following fact from linear algebra: any set of three linearly independent vectors in $\mathbb{R}^3$, spans $\mathbb{R}^3$, Problem $\boxed{6}$ b) exercise set 2, and the Frenet frame; compare the matrix in the Frenet-Serret formula with the matrix which appears in the problem statement.

Problem $\boxed{8}$

Let

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Note that f is not continuous at $(0,0)$. Show, however, that $f_x(0,0)$ and $f_y(0,0)$ both exist. Hence, the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.

Problem $\boxed{9}$

Given the parametrized curve

$$\mathbf{r}(t) = \Big(2\cos(t), \sqrt{2}\sin(t), \sqrt{2}\sin(t)\Big), \quad t \in [0, 2\pi],$$

a) Find the velocity, the speed, and the acceleration.

b) Find the arclength of the curve.

c) Show that the curve lies on a sphere centred at the origin. What is the radius of this sphere?

d) Show that curve lies in the plane $y - z = 0$.

Problem $\boxed{10}$

If $x = e^s \cos t$, $y = e^s \sin t$, and $z = u(x, y) = v(s, t)$, show that

$$\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = (x^2 + y^2)\Big(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\Big).$$