MA1103 Exercise Set 3

Norwegian University of Science and Technology

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IMPORTANT: The deadline for this exercise is Friday Feb 5. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

Problem 1

Compute $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(A+B)$ for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 2

Determine the arclength parametrization of the given curves below.

a) $c(t) = (t, 2t, 3t), \quad 0 \le t \le 12.$

b) $c(t) = (t^2, t^3), \quad 0 \le t < \infty.$

c) $c(t) = (e^t, e^{-t}, \sqrt{2}t), \quad 0 \le t < \infty.$

Problem 3

Find the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ for the curve (1) at the point indicated.

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k} \qquad \text{at} \quad (1, 1, 2), \tag{1}$$

where $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 1), \mathbf{k} = (0, 0, 1)$.

Problem 4

Let $\mathbf{u}, \mathbf{v}$ be two given vectors in a (real) vector space V equipped with an inner product $\langle , \rangle_V$, and let $S(\mathbf{v})$ denote the (real) linear span of $\mathbf{v}$. The **projection of u on v**, $P_{\mathbf{v}}(\mathbf{u})$, is the point in $S(\mathbf{v})$ closest to $\mathbf{u}$.

a) Determine a formula for $P_{\mathbf{v}}(\mathbf{u})$.

b) Find $P_{(1,0,-2)}(1, 2, 4)$.

Problem 5

Specify the natural domain of the function below.

$$f(x, y) = \ln(1 + xy).$$

Problem 6

Find all first order partial derivatives of the given functions below.

a) $f(x, y) = 4x + 5y$.

b) $f(x, y) = 8x^2y + 2x - 5y + 3$.

c) $f(x, y) = \frac{\sin(xy)}{x^2+y^2}$.

d) $f(\mathbf{x}) = e^{|\mathbf{x}|^2}$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $|\mathbf{x}|^2 = x_1^2 + \cdots + x_n^2$.

Problem 7

How can the function

$$f(x, y) = \frac{x^2 + y^2 - x^3y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0),$$

be defined at point $(0, 0)$ so that it becomes continuous at all points of the xy-plane?

Problem 8

Determine in each case below whether the limit exists, and in the case it does, give its value.

a) $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$.

b) $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy}$.

c) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Problem 9

Define a function

$$f(x,y) = \begin{cases} (x^3+y)\sin\dfrac{1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(1) Determine $f_x(0,0)$ and $f_y(0,0)$ if they exist.

(2) Calculate $f_x(x,y)$. Is $f_x(x,y)$ continuous at $(0,0)$?

Problem 10

a)

$U \subseteq \mathbb{R}^n$ is called **open** if for all $p \in U$ there is $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U$, where $B_\varepsilon(p)$ is the open ball centred at p with radius ε given as the set $\{x \in \mathbb{R}^n : |x-p| < \varepsilon\}$. Show that $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the following holds: for all $V \subseteq \mathbb{R}^m$ open, $\mathbf{F}^{-1}(V)$ is open in $\mathbb{R}^n$, where $\mathbf{F}^{-1}(V)$ denotes the preimage of V under $\mathbf{F}$ given as the set $\{x \in \mathbb{R}^n : \mathbf{F}(x) \in V\}$.

b)

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{G} : \mathbb{R}^m \to \mathbb{R}^p$ be two continuous maps. Using a), show that the composite map $\mathbf{G} \circ \mathbf{F}$ is continuous. **Hint:** show that for any $W \subseteq \mathbb{R}^p$, $(\mathbf{G} \circ \mathbf{F})^{-1}(W) = \mathbf{F}^{-1}(\mathbf{G}^{-1}(W))$.

References

[1] R.A. Adams, C. Essex. *Calculus, A Complete Course*, Ninth edition, Pearson.