MA1103 Exercise Set 2

Norwegian University of Science and Technology

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IMPORTANT: The deadline for this exercise is Jan 29. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

Problem $\boxed{1}$

Recall that the sphere of radius $R > 0$ and center (x_0, y_0, z_0) in $\mathbb{R}^3$ is given by the set of points $(x, y, z) \in \mathbb{R}^3$ whose distance to the point (x_0, y_0, z_0) is *exactly* equal to R. Determine an equation in (x, y, z) for the sphere S^2 centred at $(1, 1, 1)$ and with radius $r > 0$. Decide on a point $p \in S^2$ (there are many choices and you are free to choose) in the case $r = 1$. Determine the (equation of the) tangent plane to S^2 at the point p.

Problem $\boxed{2}$

Evaluate the determinant below.

$$\begin{vmatrix} 1 & 2 & -4 \\ -2 & 2 & 1 \\ -3 & 4 & -2 \end{vmatrix}$$

Problem $\boxed{3}$

An object moves along the curve $y = x^2, z = x^3$ with constant vertical speed $\frac{dz}{dt} = 3$. Find the velocity and acceleration of the object when it is at the point $(2, 4, 8)$.

Problem $\boxed{4}$

Let A be a positive semidefinite (symmetric) matrix of size $n \times n$. Prove that all its eigenvalues are non-negative.

Problem $\boxed{5}$

Parametrize the curve of intersection of the given surfaces. (*Note*: The answer is not unique.)

$$z = x^2 + y^2 \quad \text{and} \quad 2x - 4y - z - 1 = 0.$$

Problem $\boxed{6}$

a)

Prove that the curvature of any circle in the plane is constant and determine it expressed as a function of the parameters of the circle (the parameters of a circle are its radius and center).

b)

Let $\mathbf{r} = \mathbf{r}(t) \subseteq \mathbb{R}^3$ be a smooth curve. Show that if $|\mathbf{r}|$ is constant, then $\mathbf{v}$ is perpendicular to $\mathbf{r}$, where $\mathbf{v}$ denotes the velocity vector of $\mathbf{r}$.

c)

Find the unit tangent vector, unit normal vector, and curvature at a general point on the given curve

$$\mathbf{r} = \mathbf{r}(t) = e^t \left(\cos t, \sin t, 1\right).$$

Problem $\boxed{7}$

Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t \sin t, \quad y = a\sin^2 t, \quad z = bt.$$

What is the length of $\mathcal{C}$ between $t = 0$ and $t = T > 0$?

Problem $\boxed{8}$

Let $c = c(t)$ be a smooth (parametrized) curve in $\mathbb{R}^3$, $t \in [a, b]$. Show that c is parametrized by arclength if and only if $a = 0$ and $|c'(t)| = 1$.

Problem $\boxed{9}$

Show that if the curvature $\kappa(s)$ and the torsion $\tau(s)$ are both nonzero constants, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circle helix. (*Hint*: Find a helix having the given curvature and torsion.)

References

[1] R.A. Adams, C. Essex. *Calculus, A Complete Course*, Ninth edition, Pearson.