MA1103 Exercise Set 12

Norwegian University of Science and Technology

April 9, 2021

IMPORTANT: The deadline for this exercise is Friday April 23rd. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

The first 4 problems concern vector fields **in the plane**, while the last 5 problems concern vector fields in **space**.

Problem $\boxed{1}$

Evaluate $\displaystyle\int_L x^2y\,dx - xy^2\,dy$, where L is the clockwise boundary of the region $0 \le y \le \sqrt{9-x^2}$.

Problem $\boxed{2}$

Evaluate $\displaystyle\int_L \left(x\sin y^2 - y^2\right)dx + \left(x^2y\cos y^2 + 3x\right)dy$, where L is the counterclockwise boundary of the trapezoid with vertices $(0,-2),(1,-1),(1,1)$, and $(0,2)$.

Problem $\boxed{3}$

Compute the area of the region bounded by (på norsk: avgrenset av) the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ using a *line integral.*

Problem $\boxed{4}$

Let C be a simple smooth closed curve in the plane which surrounds the origin, oriented counter-clockwise, and let

$$\mathbf{F}(x,y) = \frac{1}{x^2+y^2}(-y,x), \quad (x,y) \neq (0,0).$$

Use Green's theorem to show that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 2\pi$. **Hint:** we cannot apply Green's theorem *directly* (!) Why not?

Problem $\boxed{5}$

Old exam problem. Let $\mathbf{F} : (x,y,z) \mapsto (xy^2, y, z)$. Use the divergence theorem to compute

$$\int_T \mathbf{F}\cdot\mathbf{n}\, dS,$$

where T is the part of the sphere $x^2+y^2+z^2 = 4$ with $y \geq 0$ and $\mathbf{n}$ is the outward unit normal vector of the sphere.

Problem $\boxed{6}$

Let $U \subseteq \mathbb{R}^3$ be an open subset containing a body $M \subseteq \mathbb{R}^3$ whose boundary ∂M is a smooth positively oriented surface. Let ν denote the unit normal vector field along ∂M and suppose that u, v are smooth real-valued functions defined on u. Prove the following so-called *Green's formulae*:

(i)

$$\iiint_M \Delta u\, dV = \iint_{\partial M} \frac{\partial u}{\partial \nu}\, dS.$$

(ii)

$$\iiint_M \nabla u \cdot \nabla v\, dV = \iint_{\partial M} u\frac{\partial v}{\partial \nu}\, dS - \iiint_M u\Delta v.$$

(iii)

$$\iiint_M (u\Delta v - v\Delta u)\, dV = \iint_{\partial M} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)\, dS.$$

Problem $\boxed{7}$

Old exam problem. Let S be the cylinder $\{(x, y, z) : x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$. Determine

$$\iint_S \operatorname{curl}(\mathbf{v}) \cdot \mathbf{n}\, dS$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is a C^2-differentiable vector field on $\mathbb{R}^3$ with v_1, v_2 independent of the z-variable.

Problem $\boxed{8}$

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $\mathbf{c}$ be a constant vector. Show that $\nabla \bullet (\mathbf{c} \times \mathbf{r}) = 0$, $\nabla \times (\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}$, and $\nabla(\mathbf{c} \bullet \mathbf{r}) = \mathbf{c}$.

Problem $\boxed{9}$

Let L be the curve given by the intersection of the two surfaces $(x-1)^2+4y^2 = 16$ and $2x+y+z = 3$, oriented counterclockwise when viewed from high on the z-axis. Let

$$\mathbf{F} = (z^2 + y^2 + \sin x^2)\mathbf{i} + (2xy + z)\mathbf{j} + (xz + 2yz)\mathbf{k}.$$

Evaluate $\displaystyle\int_L \mathbf{F} \bullet\, d\mathbf{r}$.