MA1103 Exercise Set 11

Norwegian University of Science and Technology

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IMPORTANT: The deadline for this exercise is Friday April 16th. You may write solutions in Norwegian or English, as preferable. You may cooperate, but should be able to explain your solutions and reasoning in short oral presentations.

Problem $\boxed{1}$

Find

$$\iint_D x\,dS$$

where D is the the part of the parabolic cylinder $z = \frac{x^2}{2}$ that lies inside the first octant and the cylinder $x^2 + y^2 = 1$.

Problem $\boxed{2}$

Old exam problem. Given $g : D \to \mathbb{R}^2$, where

$$D = \left\{(x, y) : 1 \leq x \leq 9,\ -3\sqrt{x} \leq y \leq 3\sqrt{x}\right\},$$

and $g(x, y) = 1 + \frac{y^2}{2x}$, determine the area of the surface S given by the graph of g.

Problem $\boxed{3}$

Find the flux of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ downward through the part of the plane $x + 2y + 3z = 6$ lying in the first octant.

Problem $\boxed{4}$

In spherical coordinates (ρ, ϕ, θ), determine the area of the surface given by $\rho = 1 + \cos\phi$ for $(\phi, \theta) \in [0, \pi] \times [0, 2\pi]$.

Problem $\boxed{5}$

Evaluate

$$\iint_P \frac{dS}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},$$

where P is the plane with equation $Ax + By + Cz = D$, $D \neq 0$.

Problem $\boxed{6}$

(a) Let S be an oriented surface with continuous unit normal vector $\mathbf{n}$, and let $\mathbf{F}$ be the vector field $\mathbf{F} = \mathbf{n}$. Prove that the flux of $\mathbf{F}$ through S is equal to the area of S.

(b) Write $\mathbf{F} = (F_1, F_2, F_3)$ and $\nabla \cdot \mathbf{F} := \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. Consider in (a) the particular example that S is the unit sphere and $\mathbf{n}$ its outward-pointing unit normal. As in (a), let $\mathbf{F} = \mathbf{n}$. Conclude that in this special case, the following identity holds:

$$\iiint_{B_1} \nabla \cdot \mathbf{F} \, dV = \iint_{\partial B_1} \mathbf{F} \cdot \mathbf{n} \, dS = \Phi_{\partial B_1}(\mathbf{F}),$$

where $B_1 = \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| \leq 1\}, \partial B_1$ denotes its boundary (på norsk: randen), and $\Phi_{\mathcal{S}}(\mathbf{F})$ denotes denotes the flux of $\mathbf{F}$ *out* of a closed oriented surface $\mathcal{S}$.

Problem $\boxed{7}$

Find the total charge on the surface

$$\mathbf{r} = e^u \cos v \mathbf{i} + e^u \sin v \mathbf{j} + u\mathbf{k}, \quad (0 \leq u \leq 1, \, 0 \leq v \leq \pi),$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Problem $\boxed{8}$

Let $\mathbf{r}$ denote the identity map $\mathbb{R}^3 \to \mathbb{R}^3$ and let $\mathbf{F} = m\frac{\mathbf{r}}{|\mathbf{r}|^3}$. Determine the total flux of $\mathbf{F}$ *out* of the cube $[-a, a] \times [-a, a] \times [-a, a]$, $a > 0$. Below is an outline of steps which might be helpful:

(i) Given that the identity map $\mathbb{R}^n \to \mathbb{R}^n$ is given by $\mathbf{x} \mapsto \mathbf{x}$, for $(x, y, z) \in \mathbb{R}^3$, write an explicit expression for $\mathbf{F}(x, y, z)$.

(ii) Argue that the total flux of the cube is a sum of 6 flux contributions out of 6 surfaces, and identify these.

(iii) Determine as an integral formula, the flux of $\mathbf{F}$ using the expression in (i) out of the surfaces in (ii). **Hint:** at this point one might want to consider using symmetry to reduce work.

(iv) Given the integral formula

$$\int_{-a}^{a} \int_{-a}^{a} \frac{1}{\sqrt{a^2 + u^2 + v^2}^3} \, du \, dv = \frac{2\pi}{3a},$$

compute from (iii) the total flux out of the cube of $\mathbf{F}$.

Problem $\boxed{9}$

Determine the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

in two ways:

(a) Using double integrals, pretending you never learnt about triple integrals.

(b) Using triple integrals.

Comment on the special case that $a^2 = b^2 = c^2$.