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Solution to the exam 23 May 2017

Problem 1

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

a/ i/ <u>f is cont at $(0,0)$:</u>

$$f \text{ is cont at } (0,0) \overset{\text{def}}{\iff} \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} \overset{\text{pol coord}}{=} \lim_{r\to 0} \frac{r^3\cos\theta\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0} r\cos\theta\sin^2\theta = 0 = f(0,0)$$

$\Rightarrow$ f is cont. at $(0,0)$

ii/ <u>$\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist</u>

$$\frac{\partial f}{\partial x}(0,0) \overset{\text{def}}{=} \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{0}{h^2} - 0\right) = 0$$

and similar $\frac{\partial f}{\partial y}(0,0) = 0$

<u>f is not diff at $(0,0)$:</u>

f is diff at $(0,0)$ $\iff$ $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ AND

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - \nabla f(0,0)\cdot(x,y)|}{\|(x,y)\|} = 0$$

We know already that $\nabla f(0,0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)(0,0) = (0,0)$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - \nabla f(0,0)(x,y)|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{|x|y^2}{(x^2+y^2)^{3/2}}$$

$$\overset{\text{polar coord.}}{=} \lim_{r\to 0} \frac{r^3|\cos\theta|\sin^2\theta}{(r^2)^{3/2}} = \lim_{r\to 0} |\cos\theta|\sin^2\theta$$

But unless $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$: $\lim_{r\to 0} \cos\theta\sin^2\theta \neq 0$

$\Rightarrow$ f is not diff at $(0,0)$

b/ Let $g(t) = (at, bt)$, $a,b \neq 0$

$$\Rightarrow f\circ g(t) = \frac{ab^2t^3}{t^2(a^2+b^2)} = t\,\frac{ab^2}{a^2+b^2}$$ is a diff function and

$$(f\circ g)'(0) = \frac{ab^2}{a^2+b^2} \neq 0$$

But $\nabla f(0,0)\, g'(0) = (0,0)\cdot(a,b) = 0$

This is not a contradiction to the chain rule, because f is not diff. at $(0,0)$

<u>Problem 2</u> $r_A(t) = (t, t^2, t^2 - t)$ $r_B(t) = (8-t, t^2-4, t^2-2t)$

The distance between the space ships : $\| r_A(t) - r_B(t) \|$

Consider the „square" distance $d(t) := \|r_A(t) - r_B(t)\|^2 = \|(2t-8, 4, t)\|^2$

$$= (2t-8)^2 + 4^2 + t^2 = 3t^2 - 32t + 80$$

$d'(t) = 10t - 32 = 0 \Leftrightarrow t = \frac{16}{5}$

$d''(t) = 10 > 0 \ \forall t \rightarrow t = \frac{16}{5}$ is a loc minimum and

$d\left(\frac{16}{5}\right) = \frac{144}{5}$

Note that $d(0) = 80 > d(\frac{16}{5})$ and $\lim_{t \to \infty} d(t) = \infty$

$\Rightarrow$ The space ships have the closest distance at time $t = \frac{16}{5}$

Problem 3

a/ Find and classify all critical points of $f(x,y) = x^3 - y^2 - 3x - 6y - 1$

$\nabla f(x,y) = (3x^2 - 3, -2y - 6) = (0,0)$

$\Leftrightarrow \quad x^2 = 1$ and $y = -3$

$\Rightarrow$ two critical points $(1,-3)$ and $(-1,-3)$

$$H_f(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix}$$

$H_f(1,-3) = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det H_f(1,-3) = -12 < 0$

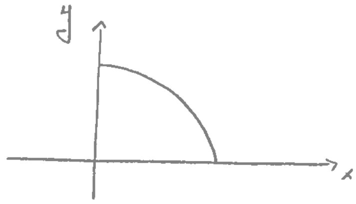
$\Rightarrow (1,-3)$ is a saddle point

$H_f(-1,-3) = \begin{pmatrix} -6 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det H_f(-1,-3) = 12 > 0$ and $\frac{\partial^2 f}{\partial x^2}(-1,-3) = -6 < 0$

$\Rightarrow (-1,-3)$ is a local maximum

b/ Find the minimum and maximum values of $f(x,y) = x^2 - xy + y^2$

on the quater circle $x^2 + y^2 = 1, \quad x,y \geq 0$



Set $g(x,y) = x^2 + y^2$, then (x,y) is a

critical point of f under the constraint $g(x,y) = 1$ if $\exists \lambda \in \mathbb{R}$:

$\nabla f(x,y) = \lambda \nabla g(x,y)$ or $\nabla g(x,y) = (0,0)$

$\nabla f(x,y) = (2x-y,\ 2y-x)$ $\qquad$ $\nabla g(x,y) = (2x,\ 2y)$

Note that $\nabla g(x,y) = (0,0) \iff x = y = 0$, but $g(0,0) \neq 1$

$\Rightarrow$ (x,y) is a critical point of f under the constraint $g(x,y) = 1$ if

$\exists\ \lambda \in \mathbb{R}$: $\quad \nabla f(x,y) = \lambda \nabla g(x,y)$

$$(2x-y,\ 2y-x) = \lambda (2x,\ 2y)$$

$\iff$ $\quad 2x - y = 2\lambda x \quad$ and $\quad 2y - x = 2\lambda y$

$\iff$ $\quad 2x(1-\lambda) = y \quad$ and $\quad 2y(1-\lambda) = x$

$\Rightarrow$ $\quad 2xy(1-\lambda) = y^2 \quad$ and $\quad 2yx(1-\lambda) = x^2$

$\Rightarrow$ $\quad y^2 = x^2 \quad \overset{g}{\Rightarrow} \quad 2x^2 = 1$

$\Rightarrow$ $\quad x = \pm \frac{1}{\sqrt{2}} \quad$ and $\quad y = \pm \frac{1}{\sqrt{2}}$

Due to the condition $x, y \geq 0$, we deduce that $(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the only critical point.

The set $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1,\ x \geq 0,\ y \geq 0\}$ is a closed and bounded set, f is continuous $\Rightarrow$ f attains both, its maximum and its minimum on this set!

At the boundaries of the quarter circle we have

$$f(1,0) = 1 \quad \text{and} \quad f(0,1) = 1$$

At the critical point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ we have that $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$\Rightarrow$ f attains its maximum at $(1,0)$ and $(0,1)$ and

f attains its minimum at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

<u>Alternatively</u> Notice that the quater circle can be parameterized

by $\quad r(t) = (\cos t, \sin t) \qquad t \in [0, \frac{\pi}{2}]$

Find the maximum and minimum of $f(r(t))$:

$$\frac{d}{dt} f(r(t)) = \left(\cos^2 t - \cos t \sin t + \sin^2 t\right)'$$

$$= -2\cos t \sin t - \cos^2 t + \sin^2 t + 2 \sin t \cos t$$

$$= \sin^2 t - \cos^2 t = 0$$

$$\Leftrightarrow \quad \sin^2 t = \cos^2 t \quad \overset{t \in [0, \frac{\pi}{2}]}{\Rightarrow} \quad t = \frac{\pi}{4}$$

$$\frac{d^2}{dt^2} f(r(t)) = 4 \sin t \cos t \quad \Rightarrow \quad \frac{d^2}{dt^2} f\left(r\left(\frac{\pi}{4}\right)\right) = 2 > 0$$

$\Rightarrow$ f attains a local minimum at $r\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

At the boundaries we have that $f(r(0)) = f(1,0) = 1$ and

$f(r(\frac{\pi}{2})) = f(0,1) = 1$

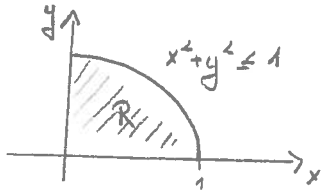
Since f is continuous and the quater circle is a bounded and closed set, f attains its maximum and minimum on this set

$\Rightarrow \quad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the minimum point with $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

$(1,0)$ and $(0,1)$ are the maximum points with $f(1,0) = f(0,1) = 1$

Problem 4

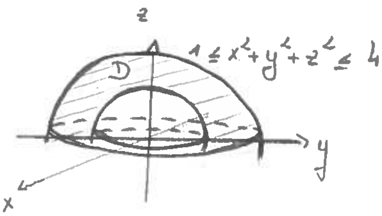
a/ $$\iint_R \frac{x}{x^2+y^2}\, d(x,y)$$



$$\iint_R \frac{x}{x^2+y^2}\, d(x,y) \underset{\text{coord}}{\overset{\text{polar}}{=}} \int_0^1 \int_0^{\pi/2} \frac{r\cos\theta}{r^2} \cdot r \, d\theta\, dr$$

$$= \int_0^1 \int_0^{\pi/2} \cos\theta \, d\theta\, dr = \sin\theta \Big|_0^{\pi/2} = 1$$

b/ $$\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, d(x,y,z)$$



$z \ge 0$

$$\iiint_D e^{(x^2+y^2+z^2)^{3/2}}\, d(x,y,z) \underset{\text{coord}}{\overset{\text{spherical}}{=}} \int_0^{2\pi}\int_0^{\pi/2}\int_1^2 e^{(r^2)^{3/2}}\, r^2 \sin\phi \, dr\, d\phi\, d\theta$$

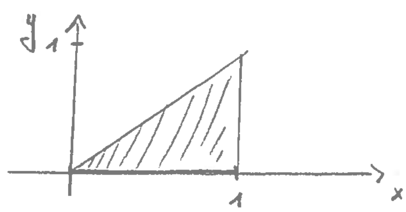
$$= 2\pi \int_0^{\pi/2} \frac{1}{3} e^{r^3} \sin\phi \Big|_{r=1}^{2} d\phi$$

$$= 2\pi \int_0^{\pi/2} \frac{1}{3}\left(e^8 - e\right) \sin\phi \, d\phi$$

$$= \frac{2}{3}\pi \left(e^8 - e\right)$$

c/ $$\int_0^1 \int_0^x f(x,y)\, dy\, dx$$

sketch the region of integration



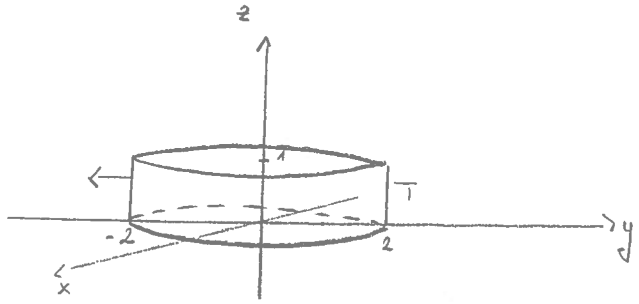
$x \in [0,1]$ $y \in [0,x]$

change the order of integration : $y \in [0,1]$ $x \in [y,1]$

$$\Rightarrow \int_0^1 \int_0^x f(x,y)\, dy\, dx = \int_0^1 \int_y^1 f(x,y)\, dx\, dy$$

Problem 5

a) Let T be the surface given by $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=4,\ 0\le z\le 1\}$



$$F(x,y,z) = (x+\sin(z^2y),\ x^2+z,\ 1-z)$$

$$\operatorname{div} F(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1+0+(-1) = 0$$

Denote by V the region $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 4,\ 0\le z\le 1\}$

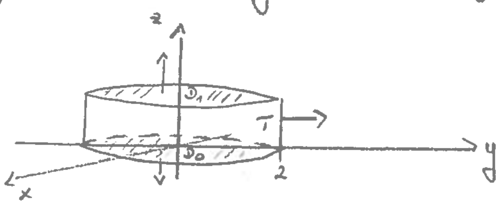
Divergence $\Rightarrow$ theorem

$$\iiint_V \operatorname{div} F \, d(x,y,z) = \iint_{\partial V} F\cdot n \, dS$$

Since $\operatorname{div} F = 0 \Rightarrow \iint_{\partial V} F\cdot n\, dS = 0$

$\partial V = T + D_0 + D_1$ where $D_0 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 4,\ z=0\}$

and $D_1 = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 4,\ z=1\}$



$$\Rightarrow \iint_T F\cdot n\, dS = -\iint_{D_0} F\cdot n\, dS - \iint_{D_1} F\cdot n\, dS$$

- $\displaystyle\iint_{D_0} F\cdot n\, dS \overset{n=(0,0,-1)}{=} -\iint_{D_0} 1-z\, dS \overset{z=0}{=} -\iint_{D_0} 1\, dS = -\operatorname{area}(D_0) = -4\pi$
- $\displaystyle\iint_{D_1} F\cdot n\, dS \overset{n=(0,0,1)}{=} \iint_{D_1} 1-z\, dS \overset{z=1}{=} \iint_{D_1} 0\, dS = 0$

$$\Rightarrow \underline{\underline{\iint_T F\cdot n\, dS = 4\pi}}$$

b/ Let S be a sphere in $\mathbb{R}^3$ and F a smooth vector field in $\mathbb{R}^3$

We have to show that $\iint_S \operatorname{curl} F \cdot n \, dS = 0$

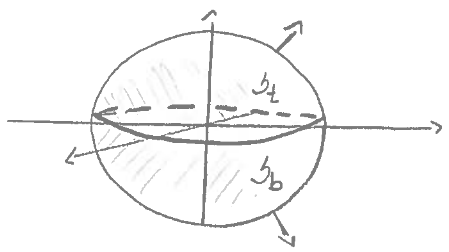
Recall that $\operatorname{div}(\operatorname{curl} F) = 0$. Set $G = \operatorname{curl} F$ and denote by V the region in $\mathbb{R}^3$ enclosed by the sphere S

Divergence theorem $\Rightarrow$ $\iiint_V \operatorname{div} G \, d(x,y,z) = \iint_S G \cdot n \, dS$

Since $\operatorname{div} G = \operatorname{div}(\operatorname{curl} F) = 0$, we obtain that

$$\iint_S \operatorname{curl} F \cdot n \, dS = \iint_S G \cdot n \, dS = \iiint_V \operatorname{div} G \, d(x,y,z) = 0$$

Alternatively Consider the sphere S



S_t : upper half sphere

S_b : lower half sphere

$$\iint_S \operatorname{curl} F \cdot n \, dS = \iint_{S_b} \operatorname{curl} F \cdot n \, dS + \iint_{S_t} \operatorname{curl} F \cdot n \, dS$$

$$\overset{\text{Stokes}}{=} \int_{\partial S_b} F \cdot dS + \int_{\partial S_t} F \cdot dS$$

where ∂S_b and ∂S_t are the boundaries of S_b and S_t, respectively

Notice that ∂S_b and ∂S_t are the same curve in $\mathbb{R}^3$, but with

<u>opposite orientation</u> (!) $\Rightarrow$ $\int_{\partial S_b} F \cdot dS = - \int_{\partial S_t} F \cdot dS$

$$\Rightarrow \iint_S \operatorname{curl} F \cdot n \, dS = 0$$