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Solution Exam 07 August 2017

Problem 1

$$f(x,y) = \begin{cases} \dfrac{x^3 - x^2 y + 4xy^2}{3x^2 y + y^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

If f is cont. at $(0,0)$, then $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, but

$$\lim_{(x,x)\to(0,0)} f(x,x) = \lim_{(x,x)\to(0,0)} \frac{x^3 - x^3 + 4x^3}{3x^3 + x^3} = 1 \neq 0 = f(0,0)$$

$\Rightarrow$ f is not cont. at $(0,0)$.

Problem 2

$$r(t) = (e^t \cos t,\ e^t \sin t,\ 1) \qquad t \geq 0$$

a) Velocity vector: $r'(t) = (e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ 0)$

$$= e^t (\cos t - \sin t,\ \sin t + \cos t,\ 0)$$

$\Rightarrow$ $r'(0) = (1, 1, 0)$

Speed: $\|r'(t)\| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}$

$$= \sqrt{2}\, e^t$$

$\Rightarrow$ $\|r'(0)\| = \sqrt{2}$

Acceleration: $r''(t) = e^t r'(t) + e^t(-\sin t - \cos t,\ \cos t - \sin t,\ 0)$

$$= 2e^t(-\sin t,\ \cos t, 0)$$

$\Rightarrow$ $r''(0) = 2(0, 1, 0)$

b/ $$\kappa(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$$

$$r'(t) \overset{a)}{=} e^t (\cos t - \sin t ,\ \sin t + \cos t ,\ 0)$$

$$r''(t) \overset{a)}{=} 2e^t (-\sin t ,\ \cos t ,\ 0)$$

$$\Rightarrow \quad r'(t) \times r''(t) = \begin{vmatrix} e_1 & e_2 & e_3 \\ e^t(\cos t - \sin t) & e^t(\sin t + \cos t) & 0 \\ -2e^t \sin t & 2e^t \cos t & 0 \end{vmatrix}$$

$$= (0, 0, 2e^{2t}(\cos^2 t - \sin t \cos t + \sin^2 t + \sin t \cos t))$$

$$= 2e^{2t}(0, 0, 1)$$

$$\Rightarrow \quad \| r'(t) \times r''(t) \| = 2e^{2t}$$

$$\| r'(t) \|^3 \overset{a)}{=} (\sqrt{2} e^t)^3 = 2\sqrt{2} e^{3t}$$

$$\Rightarrow \quad \underline{\kappa(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3} = \frac{1}{\sqrt{2}} e^{-t}}$$

<u>Problem 3</u> $$f(x,y) = (x^2 + y^2) e^x$$

a/ (x,y) is a critical point $\Longleftrightarrow$ $\nabla f(x,y) = (0,0)$

$$\nabla f(x,y) = (e^x (x^2 + y^2) + 2xe^x ,\ 2ye^x) = (0,0)$$

$$\overset{e^x \neq 0}{\Longleftrightarrow} \quad x^2 + y^2 + 2x = 0 \quad \text{and} \quad 2y = 0$$

$$\Longleftrightarrow \quad y = 0 \quad \text{and} \quad (x+2)x = 0$$

$\Rightarrow$ <u>we have two crit. points : $(0,0)$ and $(-2,0)$</u>

$$Hf(x,y) = \begin{pmatrix} e^x (x^2 + y^2 + 2x + 2x + 2) & 2ye^x \\ 2ye^x & 2e^x \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \det(Hf(0,0)) = 4 > 0$$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0$$

$$\Rightarrow \underline{(0,0) \text{ loc. minimum}}$$

$$Hf(-2,0) = \begin{pmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{pmatrix} \Rightarrow \det(Hf(-2,0)) = -4e^{-4} < 0$$

$$\Rightarrow \underline{(-2,0) \text{ saddle point}}$$

b/ Tangent plane : $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$,

where $(a,b,c) = (-f_x, -f_y, 1)$ is orthogonal to $z = f(x,y)$ and $(x_0,y_0,z_0) = (0,1,1)$

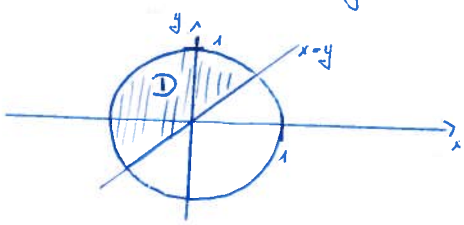
$(a,b,c) = (-f_x(0,1), -f_y(0,1), 1) = (-1,-2,1)$

Thus, the tangent plane is given by

$$-x - 2(y-1) + z - 1 = 0$$

$$\Leftrightarrow \quad \underline{z = x + 2y - 1}$$

Problem 4 $\quad D$ is given by $x^2+y^2 \leq 1$ and $y \geq x$

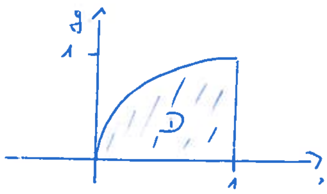


pol. coord.

$$\iint_D 2\cos(x^2+y^2)\, d(x,y) = \int_0^1 \int_{\pi/4}^{\pi/4+\pi} 2r\cos(r^2)\, d\theta\, dr = \pi \int_0^1 2r\cos(r^2)\, dr$$

$$= \pi \sin(r^2) \Big|_0^1 = \pi \sin(1)$$

Problem 5 $\quad S$ is given by $z = 2\sqrt{x}$, where $x \in [0,1]$, $y \in [0,\sqrt{x}]$



S is param. by $r : D \to \mathbb{R}^3$

$$r(x,y) = (x, y, 2\sqrt{x})$$

$$\text{area}(S) = \iint_S 1 \, dS = \iint_D \| r_x \times r_y \| \, d(x,y)$$

where $r_x \times r_y = \left(\frac{1}{\sqrt{x}}, 0, 1 \right) \Rightarrow \| r_x \times r_y \| = \sqrt{\frac{1}{x} + 1}$

$$\Rightarrow \iint_D \| r_x \times r_y \| \, d(x,y) = \int_0^1 \int_0^{\sqrt{x}} \sqrt{\frac{1}{x} + 1} \, dy \, dx$$

$$= \int_0^1 \sqrt{1+x} \, dx = \frac{2}{3} (1+x)^{3/2} \Big|_0^1 = \underline{\frac{2}{3} (2^{3/2} - 1)}$$

Problem 6

$$F(x,y,z) = (zy^2 e^{xz}, 2ye^{xz}, xy^2 e^{xz})$$

i/ Let $f(x,y,z) = y^2 e^{xz}$, then $F = \nabla f$

$\Rightarrow F$ is a conservative field

ii/ C is param by $r(t) = (\cos(2t), \sin t, t(\pi - 2t))$ $t \in [0, \pi/2]$

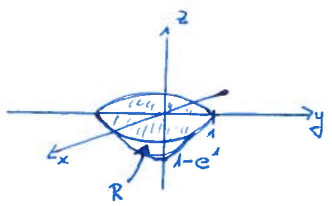
$$\int_C F \cdot dr = \int_C \nabla f \, dr = f(r(\pi/2)) - f(r(0))$$

$$= f(-1,1,0) - f(1,0,0) = 1 - 0 = \underline{1}$$

Problem 7

$$R = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, \; 1 - e^{1-x^2-y^2} \leq z \leq 0 \}$$

a/



$$\text{vol}(R) = \iiint_R 1 \, d(x,y,z) = \iint_D \int_{1-e^{1-x^2-y^2}}^{0} 1 \, dz \, d(x,y), \text{ where } D := \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$$

pol. coord

$$= \int_0^{2\pi} \int_0^1 (e^{1-r^2} - 1) \, r \, dr \, d\theta = 2\pi \left[-\frac{r^2}{2} - \frac{1}{2} e^{1-r^2} \right] \Big|_0^1$$

$$= 2\pi \left[-\frac{1}{2} - \frac{1}{2} + \frac{1}{2} e \right] = \underline{\pi (e-2)}$$

b/ $\quad F(x,y,z) = (2x + e^{yz},\ \sin(y),\ x - \cos(y)z)$

$\operatorname{div} F = 2 + \cos(y) - \cos(y) = 2$

$\iint_{\partial R} F \cdot dS \overset{\text{Gauß}}{=} \iiint_R \operatorname{div} F \, d(x,y,z) = 2 \operatorname{vol}(R) \overset{a)}{=} 2\pi(e-2)$