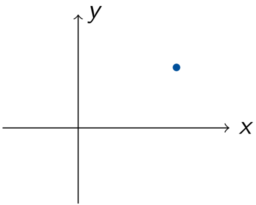
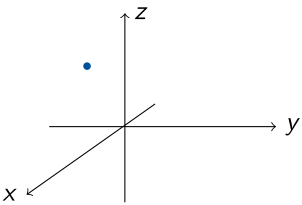
Repetition of Linear Algebra
(MA1201)

Vectors in n dimensions





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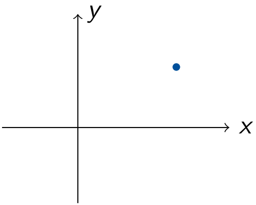

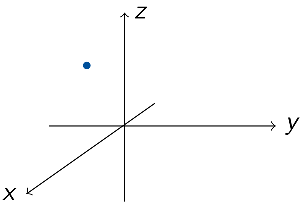

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More general:

$$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\}$$

is an n-dimensional vector space.

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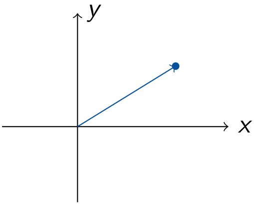

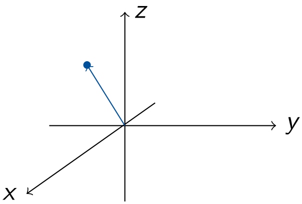

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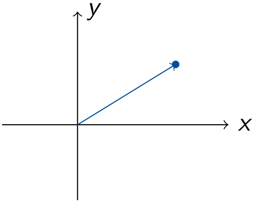

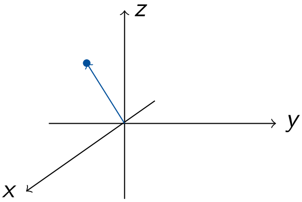

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A vector has a direction and a length.

Vectors in n dimensions

The spaces $\mathbb{R}^n$ are examples for vector spaces over $\mathbb{R}$.

Definition

$(V, +, \cdot)$ is called a vector spaces over $\mathbb{R}$, if there exists an

$$\begin{array}{lll} \text{addition} & + & : V \times V \to V \\ \text{multiplication} & \cdot & : \mathbb{R} \times V \to V \end{array}$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha, \beta \in \mathbb{R}$

A1)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	M1)	$1 \cdot \mathbf{u} = \mathbf{u}$
A2)	$\mathbf{v} + \mathbf{0}_V = \mathbf{v}$	M2)	$(\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$
A3)	$\mathbf{v} + (\text{-}\mathbf{v}) = \mathbf{0}_V$	M3)	$(\alpha\beta) \cdot \mathbf{u} = \alpha \cdot (\beta \cdot \mathbf{u})$
A4)	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	M4)	$\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{w}$

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Usually we write $\alpha\mathbf{u}$ instead of $\alpha \cdot \mathbf{u}$ for the scalar multiplication.

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Every element $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ can be written as a scalar combination of $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

Recall that $a_1, a_2, a_2 \in \mathbb{R}$.

Inner product (scalar product)

Definition

The inner product (scalar product/dot product) is a map

$$V \times V \to \mathbb{R}, \qquad (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u} \\ (\alpha \mathbf{u}) \cdot \mathbf{v} &= \alpha(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (\alpha \mathbf{v}) \\ \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{u} &\geq 0, \quad \text{where} \quad \mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = \mathbf{0} \end{aligned}$$

Alternative notation for the inner product: $\langle \mathbf{u}, \mathbf{v} \rangle$.

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The inner product of two elements $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \ldots + u_n v_n \in \mathbb{R}$$

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The norm of an element $\mathbf{a} \in V$ is defined by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

A vector $\mathbf{v} \in V$ satisfying $\|\mathbf{v}\| = 1$ is called a unit vector.

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Every vector $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \neq 0$ can be normalized to a unit vector by

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Note: The book uses the notation $|\mathbf{a}|$ for the length!

Inner product (scalar product)

Properties

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, then

- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where $\theta \in [0, \pi]$ is the angle
- $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$ (Chauchy–Schwarz inequality)
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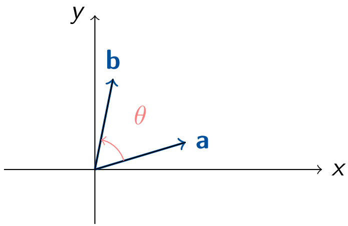

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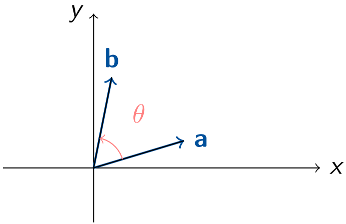

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a and **b** are orthogonal
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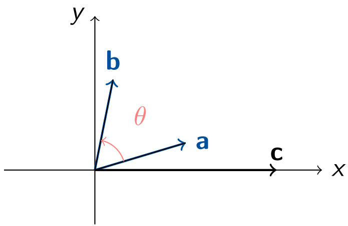

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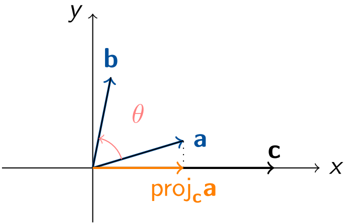

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$$\text{proj}_{\mathbf{c}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{c}}{\|\mathbf{c}\|^2}\mathbf{c}, \quad \mathbf{c} \neq 0$$

projection of **a** onto **c**.

Example

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ two vectors given by

$$\mathbf{a} = (1, 0, 3) \qquad \mathbf{b} = (-2, 1, 2)$$

- Length of $\mathbf{a}$ and $\mathbf{b}$
- Show that $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$
- Angle between $\mathbf{a}$ and $\mathbf{b}$
- Projection of $\mathbf{a}$ onto $\mathbf{b}$

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- The determinant of a (2×2)–matix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

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- If $\det(A) \neq 0$, then A has an inverse A^{-1} so that

$$AA^{-1} = A^{-1}A = \mathbb{1}_n.$$

Example

Compute the determinant of

- $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$
- $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ -1 & 2 & -4 \end{pmatrix}$

Cross product

The cross product is another product defined on $\mathbb{R}^3$.

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The cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

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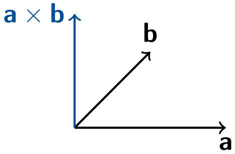

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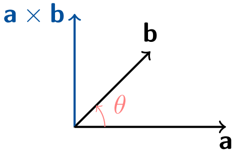
$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \in \mathbb{R}^3.$$



$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta, \qquad \theta \in [0, \pi]$$

$\mathbf{a} \times \mathbf{b} = 0$ if and only if $\theta \in \{0, \pi\}$

Cross product

The cross product is another product defined on $\mathbb{R}^3$.

Definition

The cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

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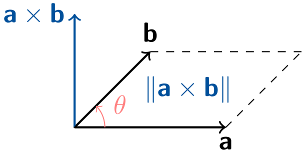

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta, \qquad \theta \in [0, \pi]$$

$\mathbf{a} \times \mathbf{b} = 0$ if and only if $\theta \in \{0, \pi\}$

Example

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be two vectors given by

$$\mathbf{a} = (3, -1, 1) \qquad \mathbf{b} = (1, 2, -1).$$

- Cross product $\mathbf{a} \times \mathbf{b}$
- Find a unit vector orthogonal on $\mathbf{a}$ and $\mathbf{b}$
- Area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$
- Angle between $\mathbf{a}$ and $\mathbf{b}$

Cross product

Properties

If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, then

- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- $\mathbf{0} \times \mathbf{a} = \mathbf{0}$
- $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Cross product

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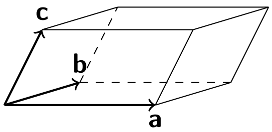
The volume of a parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is given by

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$$



Example

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be vectors given by

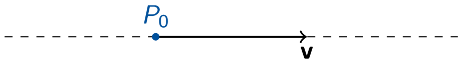
$$\mathbf{a} = (1, 0, 3) \qquad \mathbf{b} = (2, 1, -2) \qquad \mathbf{c} = (5, 0, 4).$$

- Find the volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

Lines

A line through a point $P_0 \in \mathbb{R}^2$ in direction of $\mathbf{v} \in \mathbb{R}^2$ is given by

$$l := \{P_0 + t\mathbf{v} \mid t \in \mathbb{R}\}$$



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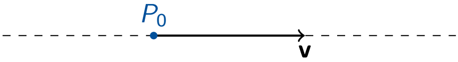
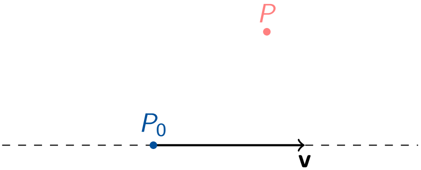


The distance between a point $P \in \mathbb{R}^2$ and a line $l = P_0 + t\mathbf{v}$ is given by

$$d(P, l) = \frac{\| \overrightarrow{P_0P} \times \mathbf{v}\|}{\|\mathbf{v}\|}$$



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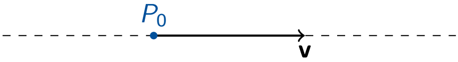

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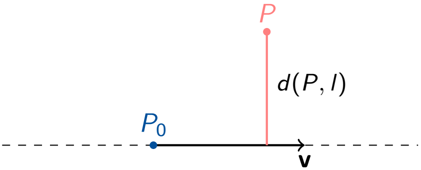

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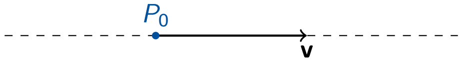

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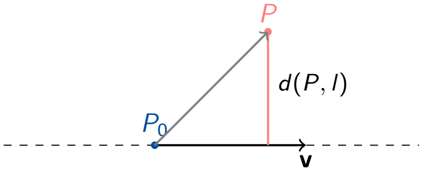

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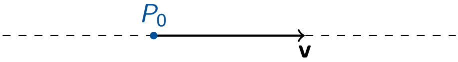

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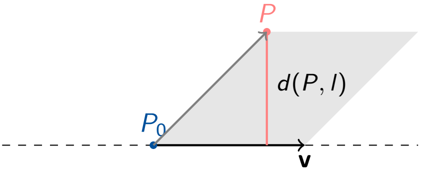

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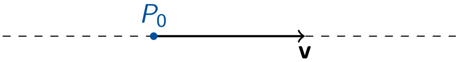

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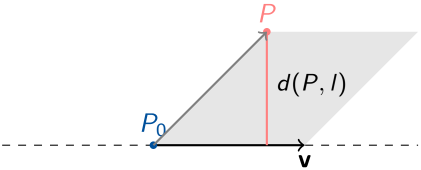
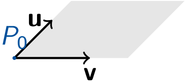


$$\|\overrightarrow{P_0P} \times \mathbf{v}\| = d(P, l)\|\mathbf{v}\|$$

Planes

A plane through a point P_0 spanned by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by

$$P := \{P_0 + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\}$$



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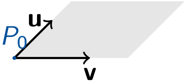
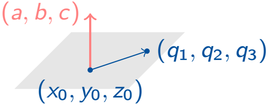


A plane through a point (x_0, y_0, z_0) being orthogonal to (a, b, c) is given by

$$P := \{(x, y, z) \mid a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\}$$



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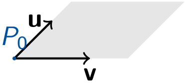

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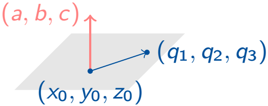

Alternatively, we can write: $P := \{(x, y, z) \mid ax + by + cz + d = 0\}$, where $d = -ax_0 - by_0 - cz_0$.

Planes

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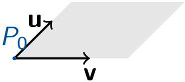

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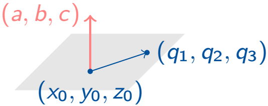

Alternatively, we can write: $P := \{(x, y, z) \mid ax + by + cz + d = 0\}$, where $d = -ax_0 - by_0 - cz_0$.
The distance between a point (x, y, z) and the plane P is given by

$$d((x, y, z), P) = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Examples

- Find an equation for the plane orthogonal to $(1, 1, 1)$ containing the point $(1, 0, 0)$.
- Find the distance from $Q = (2, 0, -1)$ to the plane given by

$$3x - 2y + 8z + 1 = 0.$$